2/5/2020 | 2.8. Subspaces

def A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ s.t.

(a) $\vec{0} \in H$

(b) $\vec{u}+\vec{v} \in H$ if $\vec{u},\vec{v} \in H$

(c) $c\vec{u} \in H$ if $\vec{u} \in H$, c a scalar

Ex: $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, then $H = \text{Span}\{\vec{v}_1,\vec{v}_2\}$ is a subspace of $\mathbb{R}^n$

Let's check: (a) $\vec{0} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 \in H$ ✓

(b) $\vec{u} = s_1\vec{v}_1 + s_2\vec{v}_2$, $\vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2$ $\Rightarrow$ $\vec{u}+\vec{v} = (s_1+t_1)\vec{v}_1 + (s_2+t_2)\vec{v}_2 \in H$ - lin. comb. of $\vec{v}_1,\vec{v}_2$ ✓

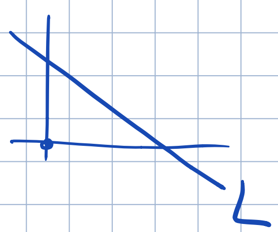
(c) $c\cdot\vec{u} = (cs_1)\vec{v}_1 + (cs_2)\vec{v}_2 \in H$

Thus: for $\vec{v}_1 \neq 0$, $\vec{v}_2 \neq c\vec{v}_1$, $\text{Span}\{\vec{v}_1,\vec{v}_2\}$ - a plane in $\mathbb{R}^n$ through $\vec{0}$

for $\vec{v}_1 \neq 0$, $\vec{v}_2 = c\vec{v}_1$, $\text{Span}\{\vec{v}_1,\vec{v}_2\}$ - a line in $\mathbb{R}^n$ through $\vec{0}$

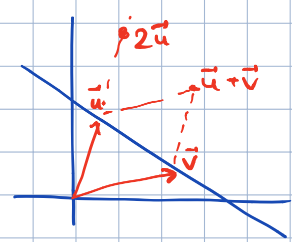
examples of subspaces

Ex: a line L not through the origin is not a subspace



(a, b, c all fail)

$\vec{0} \notin L$, $\vec{u}+\vec{v} \notin L$, $2\vec{u} \notin L$



Ex: for $\vec{v}_1,\dots,\vec{v}_p \in \mathbb{R}^n$, $H = \text{Span}\{\vec{v}_1,\dots,\vec{v}_p\}$ is a subspace of $\mathbb{R}^n$ - the subspace "spanned" (or "generated") by $\vec{v}_1,\dots,\vec{v}_p$

Ex: $\mathbb{R}^n$ itself is a subspace. Also, $H = \{0\}$ is a subspace ("zero subspace").

Column space and null space of a matrix

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def For $A = [\vec{a}_1 \dots \vec{a}_n]$ an $m\times n$ matrix, its "column space" is the set of linear comb. of columns of A

$\mathrm{Col}\,A := \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$ - subspace of $\mathbb{R}^n$

- $\mathrm{Col}\,A = \mathbb{R}^m$ iff columns of A span $\mathbb{R}^m$ $\Leftrightarrow$ pivot in each row of A.

$\mathrm{Col}\,A$ = set of all $\vec{b}$ s.t. $A\vec{x} = \vec{b}$ has a solution.

def The null space of A, $\mathrm{Nul}\,A$, is the set of all solutions of homog. eq. $A\vec{x} = \vec{0}$.

- $\mathrm{Nul}\,A$ is a subspace of $\mathbb{R}^n$
- $\mathrm{Nul}\,A$ is defined implicitly (as solutions of an eq.), $\mathrm{Col}\,A$ is defined explicitly.

Basis for a subspace

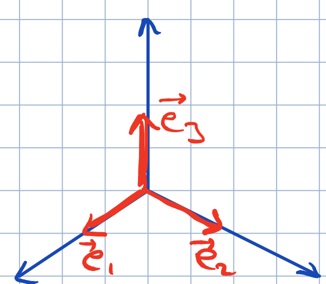
want to describe a subspace by the smallest possible set spanning it.

def A basis for a subspace H of $\mathbb{R}^n$ is a lin. indep. set in H which spans H

Ex: columns of an invertible $n\times n$ matrix A form a basis for $H = \mathbb{R}^n$ (they are lin. indep. and span $\mathbb{R}^n$ by Inv. Mat. Thm.)

E.g. For $A = I_n$, its columns $\vec{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \vec{e}_2 = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$

Set $\{\vec{e}_1, \dots, \vec{e}_n\}$ - standard basis for $\mathbb{R}^n$.



Ex: Find a basis for $\mathrm{Nul}\,A$,

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Sol: write the sol. of $A\vec{x} = 0$ in parametric form:

$$\text{Aug. Mat. } [A\ \vec{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

$$\begin{array}{cccccc} & \underbrace{x_2 \quad & & x_4 & x_5}_{\text{free}} & \end{array}$$

$x_1 - 2x_2 - x_4 + 3x_5 = 0$
$x_3 + 2x_4 - 2x_5 = 0$
$0 = 0$

$\Rightarrow$
$x_1 = 2x_2 + x_4 - 3x_5$
$x_3 = -2x_4 + 2x_5$
x_2, x_4, x_5 free

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix} = \begin{bmatrix} 2x_2+x_4-3x_5 \\ x_2 \\ -2x_4+2x_5 \\ x_4 \\ x_5\end{bmatrix} = x_2\underset{\vec{u}}{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}} + x_4 \underset{\vec{v}}{\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}} + x_5\underset{\vec{w}}{\begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}}$$

$\vec{x} = x_2\vec{u} + x_4\vec{v} + x_5\vec{w}$

Thus: $\mathrm{Nul}\, A = \mathrm{Span}\{\vec{u},\vec{v},\vec{w}\}$. Moreover, $\vec{u},\vec{v},\vec{w}$ are lin. indep.
$(x_2\vec{u}+x_4\vec{v}+x_5\vec{w}=\vec{0} \Rightarrow x_2, x_4, x_5 = 0)$

So, $\{\vec{u},\vec{v},\vec{w}\}$ - basis for Nul A.

Ex:

$$B = \underset{\vec{b}_1\ \ \vec{b}_2\ \ \vec{b}_3\ \ \vec{b}_4\ \ \vec{b}_5}{\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$ - RREF

Q: find a basis for Col B

Sol: note: $\vec{b}_2 = 2\vec{b}_1$, $\vec{b}_4 = 3\vec{b}_1 + 4\vec{b}_3$

So, any lin. comb. of $\vec{b}_1, \dots, \vec{b}_5$ is in fact a lin. comb. of $\vec{b}_1, \vec{b}_3, \vec{b}_5$ (pivot columns)

$$\vec{v} = c_1\vec{b}_1 + c_2\underbrace{\vec{b}_2}_{2\vec{b}_1} + c_3\vec{b}_3 + c_4\underbrace{\vec{b}_4}_{3\vec{b}_1+4\vec{b}_3} + c_5\vec{b}_5 = (c_1+2c_2+3c_4)\vec{b}_1 + (c_3+4c_4)\vec{b}_3 + c_5\vec{b}_5$$
$$\in \mathrm{Span}\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$$

Also, $\vec{b}_1, \vec{b}_3, \vec{b}_5$ are columns of I_4 and thus are lin. indep.
$\Rightarrow \{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ - basis for Col B.

Ex:

$$A = \underset{\vec{a}_1\ \ \vec{a}_2\ \ \vec{a}_3\ \ \vec{a}_4\ \ \vec{a}_5}{\begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ -2 & -4 & -1 & -10 & -2 \\ 3 & 6 & 0 & 9 & 1 \\ 1 & 2 & -2 & -5 & 7 \end{bmatrix}} \sim B$$

pivot columns

Q: find a basis of Col A.

Sol: A lin. dependence rel. among col. of A is a sol of $A\vec{x}=0$ and $A\vec{x}=\vec{0}$ has same solutions as $B\vec{x}=\vec{0}$!

So: $\vec{b}_2 = 2\vec{b}_1$ $\qquad\qquad$ $\vec{a}_2 = 2\vec{a}_1$
$\vec{b}_4 = 3\vec{b}_1 + 4\vec{b}_3$ $\quad \Rightarrow \quad$ $\vec{a}_4 = 3\vec{a}_1 + 4\vec{a}_3$
$\vec{b}_1, \vec{b}_3, \vec{b}_5$ lin. indep. $\qquad$ $\vec{a}_1, \vec{a}_3, \vec{a}_5$ lin. indep.

Thus: $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is a basis for Col A

<u>Thm</u> Pivot columns of A form a basis for $\operatorname{Col} A$.

<u>Warning</u>: we need pivot columns <u>of A itself</u>, not of a REF of A.