2/7/2020

LAST TIME.

Ex:

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ – RREF

$\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \vec{b}_4 \ \vec{b}_5$

Q: find a basis for Col B

Sol: note: $\vec{b}_2 = 2\vec{b}_1$, $\vec{b}_4 = 3\vec{b}_1 + 4\vec{b}_3$

So, any lin. comb. of $\vec{b}_1, \dots, \vec{b}_5$ is in fact a lin. comb. of $\vec{b}_1, \vec{b}_3, \vec{b}_5$ (pivot columns)

$$\vec{v} = c_1\vec{b}_1 + c_2\underbrace{\vec{b}_2}_{2\vec{b}_1} + c_3\vec{b}_3 + c_4\underbrace{\vec{b}_4}_{3\vec{b}_1+4\vec{b}_3} + c_5\vec{b}_5 = (c_1+2c_2+3c_4)\vec{b}_1 + (c_3+4c_4)\vec{b}_3 + c_5\vec{b}_5$$
$$\in \text{Span}\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$$

Also, $\vec{b}_1, \vec{b}_3, \vec{b}_5$ are columns of I_4 and thus are lin. indep.

$\Rightarrow \{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ – basis for Col B.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ -2 & -4 & -1 & -10 & -2 \\ 3 & 6 & 0 & 9 & 1 \\ 1 & 2 & -2 & -5 & 7 \end{bmatrix} \sim B$$

$\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4 \ \vec{a}_5$

pivot columns

Q: find a basis of Col A.

Sol: A lin. dependence rel. among col. of A is a sol of $A\vec{x} = 0$ and $A\vec{x} = \vec{0}$ has same solutions as $B\vec{x} = \vec{0}$!

So: $\vec{b}_2 = 2\vec{b}_1$ $\Rightarrow$ $\vec{a}_2 = 2\vec{a}_1$

$\vec{b}_4 = 3\vec{b}_1 + 4\vec{b}_3$ $\Rightarrow$ $\vec{a}_4 = 3\vec{a}_1 + 4\vec{a}_3$

$\vec{b}_1, \vec{b}_3, \vec{b}_5$ lin. indep. $\Rightarrow$ $\vec{a}_1, \vec{a}_3, \vec{a}_5$ lin. indep.

Thus: $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is a basis for Col A

Thm Pivot columns of A form a basis for Col A.

Warning: we need pivot columns <u>of A itself</u>, not of a REF of A.

2.9. Dimension and rank

Coordinate systems: H - subspace, $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ a basis for H

any vector $\vec{x} \in H$ can be written as a lin. comb of basis vectors in a unique way!

- if ① $\vec{x} = c_1 \vec{b}_1 + \dots + c_p \vec{b}_p$
  ② $\vec{x} = d_1 \vec{b}_1 + \dots + d_p \vec{b}_p$

two representations of $\vec{x}$ as a lin. comb $\underset{①-②}{\Rightarrow}$ $\vec{0} = (c_1 - d_1)\vec{b}_1 + \dots + (c_p - d_p)\vec{b}_p$

since $\mathcal{B}$ lin. indep., we have $c_1 = d_1, \dots, c_p = d_p$, i.e. ① = ②

def Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ a basis for H. For each $\vec{x} \in H$,

$\vec{x} = c_1 \vec{b}_1 + \dots + c_p \vec{b}_p$; $\boxed{c_1, \dots, c_p}$ - coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ - coordinate vector of $\vec{x}$ (rel. to $\mathcal{B}$), or $\mathcal{B}$-coordinate vector of $\vec{x}$.

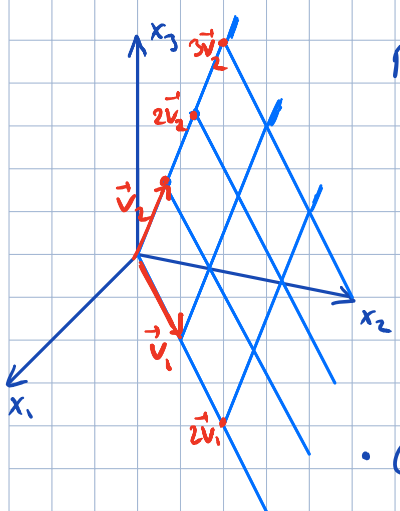
Ex: $\underbrace{\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}}_{\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} \text{ basis for } H = \text{Span}\{\vec{v}_1, \vec{v}_2\}}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 10 \\ 11 \end{bmatrix}$

Q: (a) is $\vec{x}$ in H?
(b) if yes, find $[\vec{x}]_{\mathcal{B}}$

Sol: $\vec{x} \in H$ iff eq. $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{x}$ $(*)$ is consistent

Aug. mat. $\begin{bmatrix} 2 & -1 & 1 \\ 5 & 0 & 10 \\ 1 & 3 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ Hence, $(*)$ is consistent, $c_1 = 2$, $c_2 = 3$ - solution, $\boxed{\vec{x} = 2\vec{v}_1 + 3\vec{v}_2}$

$\Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



Basis $\mathcal{B}$ determines a coordinate system on H

points in H are in $\mathbb{R}^3$ but are determined by $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^2$

Correspondence $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a one-to-one correspondence between H and $\mathbb{R}^2$ preserving linear combinations ("isomorphism")

H is "isomorphic" to $\mathbb{R}^2$.

- Generally, if $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ a basis for H, mapping $\underset{\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}}{H \to \mathbb{R}^p}$ is a 1-1 correspondence which makes H "look and act" like like $\mathbb{R}^p$.

Claim If H has a basis of p vectors, then each basis in H has exactly p vectors.

def The **dimension** of a nonzero subspace H, $\dim H$ is the number of vectors in any basis in H. Also, $\dim\{\vec{0}\} = 0$ (convention).

Ex: • $\dim \mathbb{R}^n = n$, every basis in $\mathbb{R}^n$ consists of n vectors

• $\dim$ (plane in $\mathbb{R}^3$ through $\vec{0}$) $= 2$ • $\dim$ (line in $\mathbb{R}^3$ through $\vec{0}$) $= 1$.

Ex: Nul A = basis vectors correspond to free variables of $A\vec{x} = \vec{0}$

So: $\dim \text{Nul } A = \#$ non-pivotal columns
$= \#$ free variables

def: The rank of a matrix A, rank A, is $\dim \text{Col } A$.
Thus, rank $A = \#$ pivotal columns

Ex:
$$A = \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ -2 & -4 & -1 & -10 & -2 \\ 3 & 6 & 0 & 9 & 1 \\ 1 & 2 & -2 & -5 & 7 \end{bmatrix}$$
Q: what is the rank of A?

Sol:
$$A \sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
3 pivot columns $\Rightarrow$ rank $= 3$

Note: $\dim \text{Nul } A = \#$ non-pivot col. $= 2$

Thm (the rank theorem)
If a matrix A has n columns, then $\boxed{\text{rank } A + \dim \text{Nul } A = n}$

Thm (basis thm)
Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any lin. indep. set of p vectors in H is automatically a basis for H. Also, any set of p vectors in H spanning H is a basis for H.

Invertible matrix thm (cont'd)

A $n \times n$ matrix. The following are equivalent to A being invertible:

(m) Columns of A form a basis for $\mathbb{R}^n$

(n) $\text{Col } A = \mathbb{R}^n$

(o) $\dim \text{Col } A = n$

(p) $\text{rank } A = n$

(q) $\text{Nul } A = \{\vec{0}\}$

(r) $\dim \text{Nul } A = 0$