LAST TIME:

- $A\vec{x} = \lambda \vec{x} \Rightarrow$ $\vec{x}$ - eigenvector, $\vec{x} \neq \vec{0}$; λ - eigenvalue
- $Nul(A-\lambda I)$ - λ-eigenspace
- $\det(A-\lambda I) = 0$ - characteristic equation on λ. (Its roots = eigenvalues)

- for A $n\times n$, char. eq. has n roots (counting with multiplicities). Some of them can be complex.

Similarity def. A is similar to B if there is an invertible P s.t.
$$P^{-1}AP = B \quad \left(\text{or equivalently } A = PBP^{-1}\right)$$

- $A \to P^{-1}AP$ - "similarity transformation" Note: $A \underset{\text{similar}}{\approx} B \Rightarrow B \approx A$

THM If A and B are similar, they have the same char poly and hence same e.v. (with same multiplicities)

WARNING 1. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \not\approx \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ same eigenvalues but not similar

2. similarity is not the same as row equivalence. row operations change eigenvalues.

5.3. Diagonalization.

Often can factorize $A = PDP^{-1}$ — allows to compute A^k efficiently for large k
D - diagonal matrix

Ex: $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}, D^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}, \dots, D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = PDP^{-1}, \quad P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

then: $A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$

$A^3 = PD\cancel{P^{-1}}\cancel{P}D^2P^{-1} = PD^3P^{-1}$

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$A^k = PD^kP^{-1} = \boxed{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}}$

def A is <u>diagonalizable</u> iff $A \approx D$ - diagonal matrix
i.e. if $A = PDP^{-1}$ for some diagonal D and invertible P.

THM (the diagonalization thm)
An $n \times n$ matrix A is <u>diagonalizable</u> iff A has n <u>lin. indep.</u> eigenvectors
$\vec{v}_1, \dots, \vec{v}_n$
$\lambda_1, \dots, \lambda_n$ - corresp. e.v.

Then, $A = PDP^{-1}$ with $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{bmatrix}$, $P = [\vec{v}_1 \cdots \vec{v}_n]$

Ex: $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ Q: diagonalize if possible
i.e. find P, D s.t. $A = PDP^{-1}$.

Sol. <u>Step I</u>: find eigenvalues of A.
char eq: $0 = \det(A - \lambda I) = \dots = -(\lambda-1)(\lambda+2)^2$ So: $\lambda = 1$, $\lambda = -2$ eigenvalues

<u>Step II</u> find 3 lin. indep. eigenvectors [if it fails, A cannot be diagonalized!]
since A is 3×3

basis for $\lambda = 1$ eigenspace: $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
basis for $\lambda = 2$ eigenspace: $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ - lin. ind. set

<u>Step III</u> Construct $P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

<u>Step IV</u> Construct $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ <u>Warning</u>: order of λ's in D should match the order of v's in P.

<u>Check</u>: $AP = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$, $PD = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

hooray!

Ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Q: diagonalizable?

Sol: $\lambda = 3$ the only eigenvalue. Basis for $\lambda = 3$ eigenspace $=$ basis for Nul $\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A-3I}$ $= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
$\left(\begin{array}{ccc} x_1 & x_2 & \\ \hline \end{array} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_2 = 0 \\ x_1 \text{ free} \end{array} \quad \vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$\Rightarrow$ cannot find two lin. ind. eigenvectors $\Rightarrow$ A not diagonalizable!

Thm An $n \times n$ mat. A with n distinct eigenvalues is diagonalizable

Ex: $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ e.v.: $\lambda = 1, 0, 2$ $\Rightarrow$ diagonalizable

- triang. mat.

- 3 distinct values

Case of non-distinct eigenvalues

Thm Let A be $n \times n$ mat whose distinct e.v. are $\lambda_1, \dots, \lambda_p$

$m_1, \dots, m_p$ - multiplicities

(a) for each $1 \le k \le p$, dimension d_k of λ_k-eigenspace is $\le m_k$.

(b) A is diagonalizable iff $\sum_{k=1}^{p} d_k = n$, i.e. iff (i) char. poly factors completely into linear factors

(ii) $d_k = m_k$ for each k.

(c) if A is diagonalizable and $\mathcal{B}_k$ - basis for eigenspace H_k, then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_p$ - eigenvector basis for $\mathbb{R}^n$.

Ex (Ex.6) $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$ Q: diagonalize if possible

Sol: $\lambda = 5, -3$ eigenvalues

$2, 2$ multiplicities

basis for $\lambda = 5$: $\vec{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

basis for $\lambda = -3$: $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

By THM (c), $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ lin. indep. So:

$$A = PDP^{-1}, \quad P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$