3/25/2020

LAST TIME:
- $\vec{u}\cdot\vec{v} = u_1v_1 + \dots + u_nv_n$
- $\vec{u} \underset{\text{orthog.}}{\perp} \vec{v}$ iff $\vec{u}\cdot\vec{v} = 0$
- $\text{proj}_{\vec{u}}\vec{y} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$

THM: Let $W \subset \mathbb{R}^n$ a subspace, $\{\vec{u}_1, \dots, \vec{u}_p\}$ - orthogonal basis for W.
For each $\vec{y} \in W$, weights in $\vec{y} = c_1\vec{u}_1 + \dots + c_p\vec{u}_p$ $(*)$ are: $\boxed{c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}}$

$\langle$ Idea: $\vec{y}\cdot\vec{u}_1 = c_1\vec{u}_1\cdot\vec{u}_1 + \cancel{c_2\vec{u}_2\cdot\vec{u}_1} + \dots + \cancel{c_p\vec{u}_p\cdot\vec{u}_1}$
$= c_1\vec{u}_1\cdot\vec{u}_1 \Rightarrow c_1 = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$. Similarly for c_j $\rangle$

Ex: $\underbrace{\vec{u}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}1\\-1\end{bmatrix}}_{\text{orthog. basis in } W=\mathbb{R}^2}$, $\vec{y} = \begin{bmatrix}2\\3\end{bmatrix} \Rightarrow \vec{y} = \underbrace{\dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}_{\frac{5}{2}}\vec{u}_1 + \underbrace{\dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}}_{\frac{-1}{2}}\vec{u}_2 = \frac{5}{2}\vec{u}_1 - \frac{1}{2}\vec{u}_2$

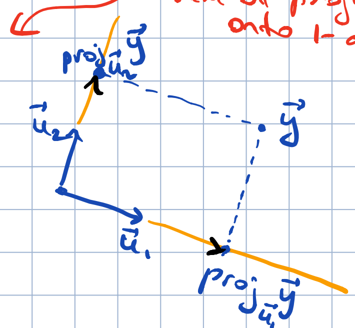
- found it without solving the lin. sys.!

- THM above implies:

$\vec{y} = \text{proj}_{\vec{u}_1}\vec{y} + \dots + \text{proj}_{\vec{u}_p}\vec{y}$

sum of projections onto 1-dim subspaces (which are mutually orthogonal)

E.g. for $W = \mathbb{R}^2$:



- works only for an orthog. basis!

Orthonormal sets $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is orthonormal if it is an orthog. set of unit vectors. If $W = \text{Span}\, S$, then S - o/n basis for W.

Ex: $\{\vec{e}_1, \dots, \vec{e}_n\}$ - o/n basis for $\mathbb{R}^n$

Ex: $\vec{v}_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$ $\{\vec{v}_1, \vec{v}_2\}$ - o/n basis for $\mathbb{R}^2$
(Obtained from Ex* by normalizing vectors to unit length, $\vec{v}_i := \frac{1}{\|\vec{u}_i\|}\vec{u}_i$)

THM An $m\times n$ matrix U has o/n columns iff $\boxed{U^TU = I}$

THM Let U be a $m\times n$ matrix with o/n columns and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then:
(a) $\|U\vec{x}\| = \|\vec{x}\|$ (b) $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$ (c) $U\vec{x}\cdot U\vec{y} = 0$ iff $\vec{x}\cdot\vec{y} = 0$

I.e. mapping $\vec{x} \mapsto U\vec{x}$ preserves length and orthogonality.

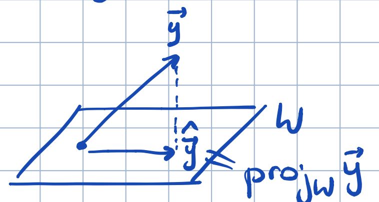
Case $m=n$: square U with o/n columns is an orthogonal matrix. U orthogonal iff $\boxed{U^{-1} = U^T}$

6.3 Orthogonal projections

Given $\vec{y} \in \mathbb{R}^n$ and $W \subset \mathbb{R}^n$, there exists a unique $\hat{y} \in W$ s.t.

(1) $\vec{y} - \hat{y} \perp W$

(2) $\hat{y}$ is the closest vector in W to $\vec{y}$.



THM (Orthogonal decomposition thm)

Let $W \subset \mathbb{R}^n$ be a subspace. Then each $\vec{y} \in \mathbb{R}^n$ can be written uniquely as $\vec{y} = \hat{y} + \vec{z}$ with $\hat{y} \in W$, $\vec{z} \in W^\perp$.

(*) Moreover, if $\{\vec{u}_1, \dots, \vec{u}_p\}$ - any orthogonal basis for W, then

$$\boxed{\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \dots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p}$$

and $\vec{z} = \vec{y} - \hat{y}$.

$\hat{y} = proj_W \vec{y}$ - orthogonal projection of $\vec{y}$ onto W.

Ex: $\underbrace{\vec{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\text{orthog. basis for } W = Span\{\vec{u}_1, \vec{u}_2\}}$; $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Q: Write $\vec{y}$ as a sum of a vector in W and a vector orthogonal to W.

Sol: $$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \underbrace{\frac{9}{30}}_{\frac{3}{10}}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \underbrace{\frac{3}{6}}_{\frac{5}{10}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}, \quad \vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$

So: $$\vec{y} = \underbrace{\begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}}_{\in W} + \underbrace{\begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}}_{\in W^\perp}$$

- formula (*) is the sum of projections of $\vec{y}$ onto lines $Span\{\vec{u}_1\}, \dots, Span\{\vec{u}_p\}$.
- if $\vec{y} \in W$, then $proj_W \vec{y} = \vec{y}$.

THM (Best approximation theorem)

Let $W \subset \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\hat{y} = proj_W \vec{y}$. Then $\hat{y}$ is the closest point in W to $\vec{y}$.

I.e. $\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$ for all $\vec{v} \in W$, $\vec{v} \neq \hat{y}$.

$\hat{y}$ is the best approximation of $\vec{y}$ by elements of W; $\|\vec{y} - \hat{y}\|$ - "error" of approximation.

Ex @: $\hat{\vec{y}} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$ - closest point to $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the plane W.

$$\underbrace{\text{dist}(\vec{y}, W)}_{\text{dist. between } \vec{y} \text{ and closest point to } \vec{y} \text{ on } W.} = \|\vec{y} - \hat{\vec{y}}\| = \|\vec{z}\| = \frac{7}{5}\sqrt{5} = \frac{7}{\sqrt{5}}$$

THM If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal basis for $W \subset \mathbb{R}^n$, then

$$\text{proj}_W \vec{y} = (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{y} \cdot \vec{u}_p)\vec{u}_p$$

If $U = [\vec{u}_1 \cdots \vec{u}_p]$, then $\boxed{\text{proj}_W \vec{y} = UU^T\vec{y}}$ for all $\vec{y} \in \mathbb{R}^n$