03/27/2020

# 6.4 Gram-Schmidt process

**Problem:** Given $W \subset \mathbb{R}^n$, find an orthogonal basis for $W$

$\phantom{W} = \mathrm{Span}\{\underbrace{\vec{x}_1, \dots, \vec{x}_p}_{\text{not orthogonal}}\}$

**Ex:** $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\quad \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad W = \mathrm{Span}\{\vec{x}_1, \vec{x}_2\} \subset \mathbb{R}^3$

Q: find an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ for $W$

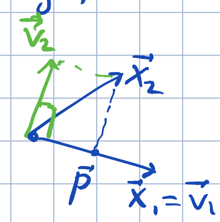
**Sol:** set $\vec{v}_1 = \vec{x}_1$.

$$\vec{x}_2 = \underbrace{\vec{p}}_{\mathrm{proj}_{\mathrm{Span}\{\vec{v}_1\}} \vec{x}_2} + \underbrace{\boxed{(\vec{x} - \vec{p})}}_{\text{orthog. to } \vec{v}_1}$$

$\vec{v}_2$

Explicitly:

$$\vec{p} = \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \frac{-1}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 0 \end{bmatrix}$$




$$\vec{v}_2 = \vec{x}_2 - \vec{p} = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

So: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix} \right\}$

- orthog. set of two vectors in $W$

$\Rightarrow$ orthog. basis for $W$.

note: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \underbrace{\vec{v}_2{}'}_{= 5 \cdot \vec{v}_2} = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} \right\}$ - also an orthog. basis for $W$.

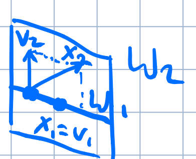
**Generally:** Let $W = \mathrm{Span}\{\underbrace{\vec{x}_1, \dots, \vec{x}_p}_{\text{basis}}\} \subset \mathbb{R}^n$ Want to construct an orthog. basis $\{\vec{v}_1, \dots, \vec{v}_p\}$ for $W$.

**Step 1:** Set $\vec{v}_1 = \vec{x}_1$, $W_1 = \mathrm{Span}\{\vec{x}_1\} = \mathrm{Span}\{\vec{v}_1\}$

**Step 2:** $W_2 = \mathrm{Span}\{\vec{x}_1, \vec{x}_2\}$. Orthog. basis: $\vec{v}_1 = \vec{x}_1$, $\vec{v}_2 = \vec{x}_2 - \mathrm{proj}_{W_1} \vec{x}_2$

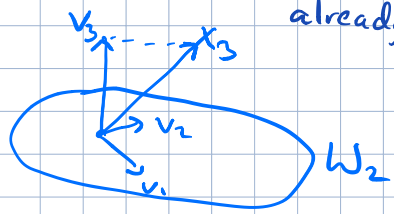
component of $\vec{x}_2$ orthog to $W_1$

$$= \boxed{\vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1}$$



**Step 3** $W_3 = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Orthog. basis: $\underbrace{\vec{v}_1, \vec{v}_2}_{\text{already constructed}}$, $\boxed{\vec{v}_3 = \vec{x}_3 - \mathrm{proj}_{W_2} \vec{x}_3}$ $= \boxed{\vec{x}_3 - \dfrac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2}$

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**Step p.** $W = W_p = \mathrm{Span}\{\vec{x}_1, \dots, \vec{x}_p\}$

orthog. basis: $\vec{v}_1, \dots, \vec{v}_{p-1}$, $\boxed{\vec{v}_p = \vec{x}_p - \mathrm{proj}_{W_{p-1}} \vec{x}_p}$

$= \boxed{\vec{x}_p - \dfrac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dots - \dfrac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}}$

**Ex:** $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ - basis for $W = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

Q: find an orthog. basis for $W$.

**Sol:** $\vec{v}_1 = \vec{x}_1 = \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$

$\vec{v}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \dfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \underset{\text{rescaling}}{\rightsquigarrow} \vec{v}_2' = \boxed{\begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}}$

$\vec{v}_3 = \vec{x}_3 - \dfrac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{x}_3 \cdot \vec{v}_2'}{\vec{v}_2' \cdot \vec{v}_2'} \vec{v}_2' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \dfrac{2}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix}$

$\underset{\text{rescaling}}{\rightsquigarrow}$ $v_3' = \boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}}$

Thus: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2' = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, v_3' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ - orthog. basis for $W$.

Q: find an orthonormal basis for $W$.

Sol: normalize $\vec{v}_1, \vec{v}_2', \vec{v}_3'$ to unit length: $\left\{ \vec{u}_1 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \dfrac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \dfrac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$

- o/n basis for $W$

## QR Factorization

**THM** (QR Factorization)

if $A$ is an $m \times n$ mat. with lin. indep. columns, then $A$ can be factored as $A = QR$ where $Q$ is an $m \times n$ mat whose columns form a o/n basis for Col $A$ and $R$ is an $n \times n$ upper triangular invertible mat. with positive diagonal entries.

Idea: $A = [\vec{x}_1 \cdots \vec{x}_n]$, $\underset{\text{Col } A}{W} = \operatorname{Span}\{\vec{x}_1, \dots, \vec{x}_n\} \subset \mathbb{R}^m$

$\Big\} $ Gram-Schmidt (+ normalization)

$\{\vec{u}_1, \dots, \vec{u}_n\}$ - o/n basis for $W$.

$$\vec{x}_k = r_{1k}\vec{u}_1 + \dots + r_{k-1\,k}\vec{u}_{k-1} + \underbrace{r_{kk}}_{\|\vec{v}_k\| > 0}\vec{u}_k + 0 \cdot \vec{u}_{k+1} + \dots + 0 \cdot \vec{u}_n$$ - from Gram-Schmidt

$$\Rightarrow \quad A = \underbrace{[\vec{u}_1 \cdots \vec{u}_n]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & r_{nn} \end{bmatrix}}_{R}$$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ find the QR factorization

Sol: $Q = [\vec{u}_1\, \vec{u}_2\, \vec{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}$

a shortcut to get R: $A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R = R$

So: $R = Q^T A = \dots = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 3/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 4/\sqrt{12} \end{bmatrix}$