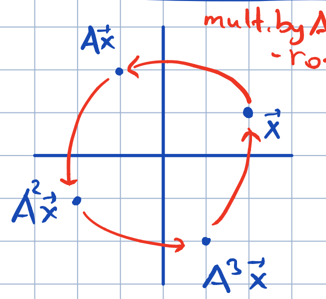
3/6/2020 | 5.5 **Complex eigenvalues.**

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



A has no eigenvectors in $\mathbb{R}^2$!

char. eq. $\lambda^2 + 1 = 0$ complex roots $\lambda = i$, $\lambda = -i$. If we allow A to act on $\underline{\underline{\mathbb{C}^2}}$:

$$\underbrace{A\begin{bmatrix} 1 \\ -i \end{bmatrix}}_{\text{eigenvector for } \lambda = i} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \underbrace{A\begin{bmatrix} 1 \\ i \end{bmatrix}}_{\text{eigenvector for } \lambda = -i} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Ex* $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ Q: find eigenvalues and a basis for each eigenspace

Sol. char. eq. $0 = \begin{vmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{vmatrix} = \lambda^2 - 1.6\lambda + 1$ solutions: $\lambda = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4}}{2}$
$= \boxed{0.8 \pm 0.6i}$

for the eigenvalue $\lambda = 0.8 - 0.6i$:

$$A - (0.8 - 0.6i)I = \begin{bmatrix} -0.3 + 0.6i & -0.6 \\ 0.75 & 0.3 + 0.6i \end{bmatrix}$$

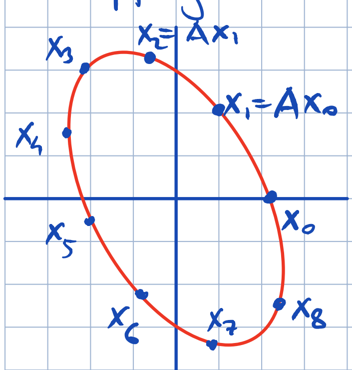
eq. for the eigenvector: $\left.\begin{array}{ll} (1) & (-0.3+0.6i)x_1 - 0.6x_2 = 0 \\ (2) & 0.75x_1 + (0.3+0.6i)x_2 = 0 \end{array}\right\}$ eq. has a nontriv sol. $\Rightarrow$ both eq. determine the same rel. between x_1, x_2

$(1) \Leftrightarrow (2) \Leftrightarrow x_1 = -(0.4 + 0.8i)x_2$

choose $x_2 = 5$ $\Rightarrow$ basis for the eigenspace: $\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

Analogously, for $\lambda = 0.8 + 0.6i$, eigenvector: $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$

- mapping $\vec{x} \longmapsto A\vec{x}$ with A above is "essentially" a rotation



Real and imaginary parts of vectors

for $\vec{x} \in \mathbb{C}^n$, complex conjugate $\overline{\vec{x}} \in \mathbb{C}^n$ - vector whose entries are complex conjugates of entries of $\vec{x}$.

Also: $\underbrace{\text{Re}\,\vec{x}}_{\text{real part of } \vec{x}}$ - vector of real parts of entries of $\vec{x}$.

$\text{Im}\,\vec{x}$ - likewise

Ex: $\vec{x} = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

Then: $\operatorname{Re}\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\operatorname{Im}\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

$\overline{\vec{x}} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix}$

for B $m\times n$ mat. with complex entries,
$\overline{B}$ - matrix with conjugate entries. We have:

$\overline{r\vec{x}} = \overline{r}\,\overline{\vec{x}}$, $\overline{B\vec{x}} = \overline{B}\,\overline{\vec{x}}$

$\overline{BC} = \overline{B}\,\overline{C}$, $\overline{rB} = \overline{r}\,\overline{B}$

Eigenvalues and eigenvectors of a real matrix that acts on $\mathbb{C}^n$

A $n\times n$ with real entries. Then $\overline{A\vec{x}} = \overline{A}\,\overline{\vec{x}} = A\,\overline{\vec{x}}$

If λ is an eigenvalue and $\vec{x} \in \mathbb{C}^n$ corresp. eigenvector, then

$A\overline{\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$.

Thus: $\overline{\lambda}$ is also an eigenvalue with $\overline{\vec{x}}$ the corresp. eigenvector.

I.e. for A real, its $\underbrace{\text{complex eigenvalues}}_{\lambda = a+ib,\ b\neq 0}$ occur in conjugate pairs.

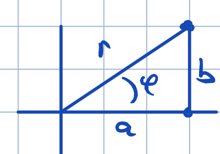
Ex: for Ex* $\quad \lambda = 0.8 - 0.6i \qquad \overline{\lambda} = 0.8 + 0.6i$ - conjugate

$\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix} = \overline{\vec{v}_1}$ - conjugate

Ex (building block for 2x2 matrices w/ cx eigenvalues)

for $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with a, b real, nonzero, eigenvalues: $\lambda = a \pm ib$ and

$C = \underbrace{r}_{|\lambda| = \sqrt{a^2+b^2}} \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scaling by } |\lambda|} \cdot \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotation by } \varphi}$



- argument of $\lambda = a+ib$

Ex: (back to Ex*)

Let $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ $\quad \lambda = 0.8 - 0.6i \quad \vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

Let $P = [\operatorname{Re}\vec{v}_1, \operatorname{Im}\vec{v}_1] = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$

And let $C = P^{-1}AP = \dots = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ - pure rotation! (by $\varphi = \arctan\frac{0.6}{0.8}$)

since $|\lambda| = \sqrt{(0.8)^2 + (0.6)^2} = 1$

Thus $A = P\underbrace{C}_{\text{rotation}}P^{-1}$

$\vec{x} = P\vec{u}$ change of variable

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, "P^{-1}"', "\text{change of variable}"] & A\vec{x} \\ \vec{u} \arrow[r, "C", "\text{rotation}"'] & C\vec{u} \arrow[u, "P", "\text{change of variable}"'] \end{tikzcd}$$

<u>Thm</u> Let A be a real 2×2 mat, with a complex eigenvalue $\lambda = a - ib$, $b \neq 0$ and $\vec{v}$ the corresp. eigenvector in $\mathbb{C}^2$. Then:

$$A = PCP^{-1} \quad \text{where} \quad P = [\operatorname{Re}\vec{v} \;\; \operatorname{Im}\vec{v}] \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$