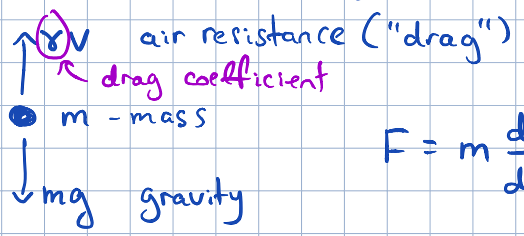
4/1/2020 | Differential Equations (Boyce, DiPrima)

1.1 Direction fields

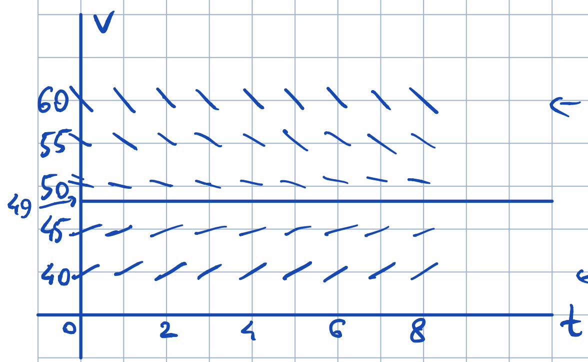
Ex 1: falling object

quantity of interest $v(t)$
velocity (in m/s)
time (in s)



$F = m\frac{dv}{dt} \rightsquigarrow \boxed{m\frac{dv}{dt} = mg - \gamma v}$ $(*)$

Set $m = 10$ kg, $\gamma = 2$ kg/s. Diff. eq. becomes: $\boxed{\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}}}$



← for $v = 60$, $\frac{dv}{dt} = -2.2$

for $v > 49$ slopes are negative $\rightsquigarrow$ speed decreasing

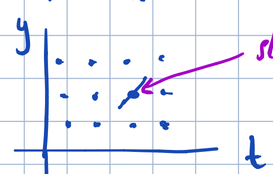
for $\boxed{v = 49}$, $\frac{dv}{dt} = 0$ - equilibrium solution

← if $v = 40$, $\frac{dv}{dt} = 1.8$ slope of the solution

for $v < 49$, slopes are positive → speed increasing

Solutions converge to $v = 49$ m/s as $t \to \infty$ - terminal velocity (or generally to $v = \frac{mg}{\gamma}$ in $(*)$)

- For diff. eq. of form $\frac{dy}{dt} = f(t,y)$ can form a direction field
  "rate function"



slope $f(t,y)$ graph of a solution passing through this point has to have this slope

Ex 2: Field mice and owls

growth proportional to current population - hypothesis

$p(t)$ population of mice in an area; time

$\frac{dp}{dt} = \boxed{rp} - k$

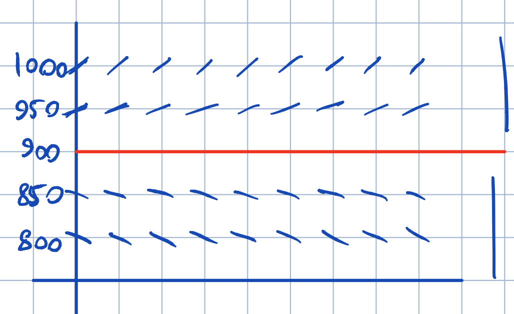
rate constant (growth rate)

predator term - mice killed by owls per unit of time

E.g. $\frac{dp}{dt} = \frac{p}{2} - 450$ in months

$r = 0.5$

$k = 15$/day $= 450$/month



$\frac{dp}{dt} > 0$ (pop. increases)

$p = 900$ equilibrium solution

$\frac{dp}{dt} < 0$ (pop. decreases)

- solutions diverge from (are repelled by) the equilibrium sol.

- sol. starting below 900 will eventually become negative - limitation of the model (unrealistic)

  >900 - p becomes huge soon.

1.2 Solutions of some differential equations

Ex 2 (cont'd) $\frac{dp}{dt} = 0.5p - 450 \quad (@) \xrightarrow[\text{rewrite}]{} \frac{dp}{dt} = \frac{p-900}{2} \longrightarrow \frac{dp/dt}{p-900} = \frac{1}{2} \quad (**)$

Note: $\frac{d}{dt} \ln|p-900| = \underbrace{\frac{1}{p-900}}_{\frac{d\ln|p-900|}{dp}} \frac{dp}{dt} = \text{l.h.s. of } (**)$

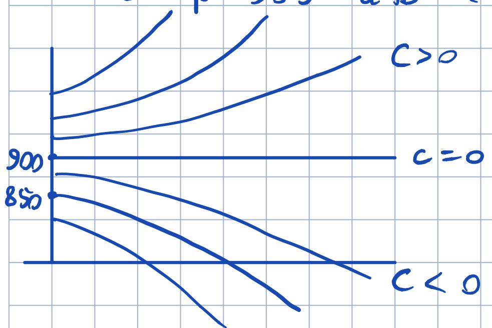
chain rule

So, (**) reads: $\frac{d}{dt}\ln|p-900| = \frac{1}{2} \xrightarrow[\text{integrate both sides}]{} \ln|p-900| = \frac{t}{2} + \underbrace{C}_{\text{arbitrary constant of integration}} \longrightarrow$

$\xrightarrow{\exp} |p-900| = e^C e^{t/2}$ or $p - 900 = \underbrace{\pm e^C}_{c} e^{t/2}$

or $\boxed{p = 900 + c\,e^{t/2}}$ (#), c arbitrary (nonzero) constant

Note: $p = 900$ also a sol. of (@), contained in if we allow $c = 0$.



- we found an infinite family of solutions, one per value of c.

graphs of solutions for different values of c

Initial value problem

$\frac{dp}{dt} = 0.5p - 450, \quad p(0) = 850$ ← initial condition

From (#) - general sol., $p(0) = 900 + c = 850 \Rightarrow c = -50 \Rightarrow \boxed{p(t) = 900 - 50e^{t/2}}$

- more generally: $\frac{dy}{dt} = ay - b, \quad y(0) = y_0$ init. cond.

diff. eq.: $\frac{dy/dt}{y - b/a} = a \Leftrightarrow \frac{d}{dt}\ln|y - \frac{b}{a}| = a \underset{\text{integrate}}{\Leftrightarrow} \ln|y - \frac{b}{a}| = at + C \rightarrow \boxed{y = \frac{b}{a} + c\,e^{at}}$ General sol.

arbitrary constant; $c=0$ - equilibrium sol.

to satisfy init. cond.: $y(0) = \frac{b}{a} + c = y_0 \xrightarrow[\text{solve for } c]{} c = y_0 - \frac{b}{a}$

$\Rightarrow \boxed{y = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}}$ (##) - sol. of the init. val. problem

General sol. produces a family of curves on (t, y) plane - "integral curves"

init. value problem - finding the integral curve passing through a given initial point.

Ex 1 (cont'd)

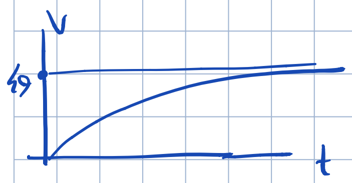
$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Assume the object is dropped ($v(0)=0$) from a height of 300 m

Q ① find its velocity at time t
② how long will it take to fall to the ground?
③ velocity at t of impact?

Sol: ① + $v(0)=0 \Rightarrow v(t) = 49 + Ce^{-t/5} \Rightarrow \boxed{v(t) = 49\left(1-e^{-t/5}\right)}$

$v(0) = 49 + C = 0 \rightarrow C = -49$



② distance the object has fallen

$$\frac{dx}{dt} = v(t) = 49\left(1-e^{-t/5}\right) \underset{\text{integrate}}{\sim} x = 49t - 245e^{-t/5} + \underbrace{k}_{\text{const of integration}}$$

$x(0)=0 \Rightarrow k = 245$

$\Rightarrow x = 49t + 245\left(1-e^{-t/5}\right)$

$$\underset{\substack{\uparrow \\ \text{time of impact}}}{x(T)} = \boxed{49T + 245\left(1-e^{-T/5}\right) = 300} \xrightarrow[\text{solve numerically for } T]{} T \approx 10.51\,s$$

③ $v(T) \approx 43.01\,m/s$ - from