2.4 Linear and nonlinear diff. equations

So far, every init. val. prob. we considered had a unique sol. (on some interval t)

THM 1 (Existence & Uniqueness of Solutions for 1st order linear equations)

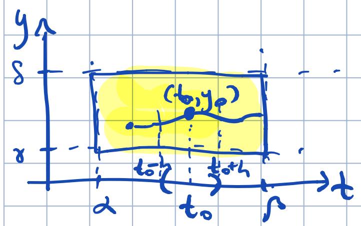
If p, g are continuous on interval I, $\alpha < t < \beta$ containing t_0, there exists a unique solution $y = \varphi(t)$ on I of the init. val. prob.
$$\boxed{\begin{aligned} y' + p(t)y &= g(t) \\ y(t_0) &= y_0 \end{aligned}}$$

[Idea: can construct the explicit sol. by integrating factors]

THM 2 (Existence & Uniqueness for 1st order non-linear equations)

Consider init. val. prob. $\boxed{y' = f(t,y), \quad y(t_0) = y_0}$ $(*)$ Assume that f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $\begin{matrix} \alpha < t < \beta \\ \gamma < y < \delta \end{matrix}$ containing (t_0, y_0).

Then a solution of $(*)$ exists and is unique in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$.



Remark
- THM2 gives a sufficient but not necessary condition for existence and uniqueness
- existence (but not uniqueness) of sol. follows just from continuity of f.

Ex: $ty' + 2y = 4t^2$, $y(1) = 2$. Q: Using THM 1, find an interval on which sol. exists and is unique.

Sol: $y' + \underbrace{\frac{2}{t}}_{p(t)} y = \underbrace{4t}_{g(t)}$.

g continuous for all t
p cont. for $t \neq 0$. I.e assumptions of THM1 hold for

two intervals: $I_1: 0 < t < \infty$ ← contains $t_0 = 1$ ⇒ Sol. exists and is unique for $0 < t < \infty$.

$I_2: -\infty < t < 0$

Note: if init. cond. were $y(-1) = 2$, interval of existence would be $-\infty < t < 0$.

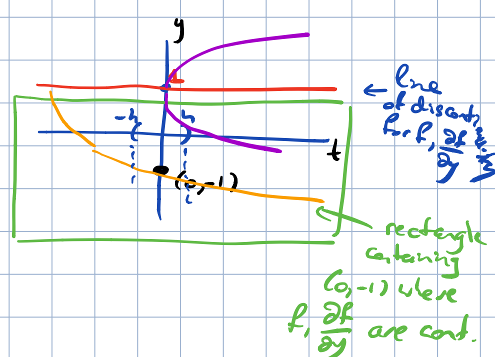
Ex: (a) $\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$ $f(x,y)$, $y(0) = -1$ apply THM 2.

(b) — " — , $y(0) = 1$

Sol: $f = \frac{3x^2+4x+2}{2(y-1)}$, $\frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$

continuous away from the line $y = 1$

(a) THM2 guarantees that sol. exists and is unique for $-h < t < h$ for some h



(in fact, from explicit sol: it exists & is unique for $-2<t<\infty$)

(b) Cannot draw a rectangle around $(0,1)$ where $f, \frac{\partial f}{\partial y}$ are cont.
So, THM2 doesn't say anything!
(In fact, from separation of variables, $y=1\pm\sqrt{x^3+2x^2+2x}$ for $x>0$ - two solutions that exist only to one side of init. cond.)

Ex 3: $\boxed{y' = y^{1/3} = f, \quad y(0)=0 \quad \text{for } t\geq 0}$ (*)   Q: apply THM2, solve init. val. prob.

Sol: $f = y^{1/3}$ continuous, $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$ discontinuous at $y=0$ (our y_0 is here)

by Remark, sol. exists in an interval around $t=0$ but possibly is not unique.

Solve (separation of var): $y^{-\frac{1}{3}}dy = dt \rightarrow \frac{3}{2}y^{\frac{2}{3}} = t + c \rightarrow y = \left(\frac{2}{3}(t+c)\right)^{3/2}$

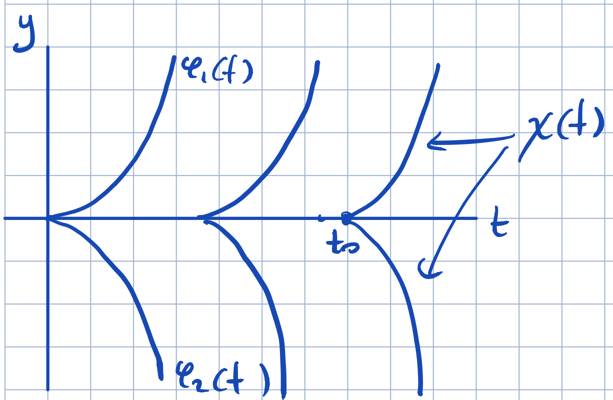
$c=0$ from init. cond.

$y = \varphi_1(t) = \left(\frac{2}{3}t\right)^{3/2}, \ t\geq 0$

$y = \varphi_2(t) = -\left(\frac{2}{3}t\right)^{3/2}, \ t\geq 0$ - two solutions of (*)

There are even more! $y = \chi(t) = \begin{cases} 0, & \text{if } 0\leq t< t_0 \\ \pm\left(\frac{2}{3}(t-t_0)\right)^{3/2}, & t\geq t_0 \end{cases}$ - differentiable everywhere, including t_0

for any $t_0 \geq 0$



- infinitely many solutions of the init. val. problem (*)

Interval of existence: $y' + p(t)y = g(t)$, $y(t_0) = y_0$ - Sol. exists and is unique on $\alpha < t < \beta$

α: nearest singularity of p or g to the left of t_0; β: — " — to the right of t_0

sol. can become singular only at values of t for which p or g is sing.

Solutions may sometimes remain continuous even at the point of discontinuity of coefficients:

$\left.\begin{aligned} y' + \frac{2}{t}y &= 4t \\ y(1) &= 1 \end{aligned}\right\} \rightarrow y = t^2$

- For a non-linear eq., interval of existence - difficult to determine.

Ex: $y' = y^2$, $y(0) = 1$ - determine the interval on which the sol. exists

Sol: $y^{-2}dy = dt \to -y^{-1} = t + c \to y = -\frac{1}{t+c}$ init. cond. $\to c = -1$

$\to y = -\frac{1}{t-1} = \frac{1}{1-t}$

interval of existence $-\infty < t < \underline{\underline{1}}$.

Note: point $t = \underline{1}$ does not seem remarkable in any way from the eq. !

If init. cond. is $y(0) = y_0$, then $c = -y_0^{-1}$ and $y = \frac{y_0}{1 - y_0 t}$
$y_0 \neq 0$

interval of existence: $-\infty < t < 1/y_0$ if $y_0 > 0$

$1/y_0 < t < \infty$ if $y_0 < 0$

linear eq. $y' + p(t)y = g(t)$	non-linear eq.
there is a general sol. depending on C - constant	might be exceptional solutions
Sol. is given explicitly, $y = \dots$	implicit solution $F(t,y) = 0$ <separable case>
possible points of discontinuity of the sol. can be identified by finding the points of disc. of the coefficients	$\leftarrow$ cannot be !