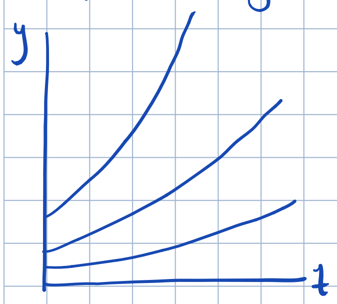
4/20/2020 | 2.5 Autonomous equations and population dynamics

$\frac{dy}{dt} = f(y)$ – autonomous eq. (1^{st} order ODE with t not appearing explicitly)

• Exponential growth $\frac{dy}{dt} = r y$, $y(0) = y_0$ $\longrightarrow$ $\boxed{y(t) = y_0 e^{rt}}$ population grows exponentially with time

r: rate of growth (if $r>0$; rate of decline if $r<0$)



Can be accurate under ideal conditions, for a limited period of time.

• Logistic growth Idea: growth rate depends on current population

$\frac{dy}{dt} = h(y)\, y$ $\qquad h(y) \approx r > 0$ for y small, $\quad h(y) < 0$ for y sufficiently large

model: $h = r - ay$, i.e.

$\boxed{\frac{dy}{dt} = (r-ay)y}$ Verhulst eq. or logistic growth

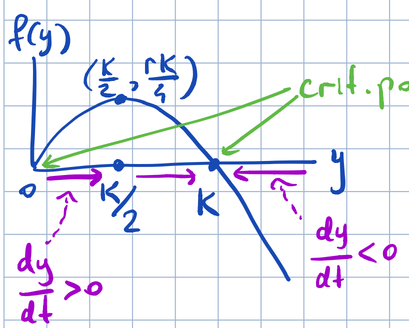
(*) or: $\boxed{\frac{dy}{dt} = \underbrace{r\left(1 - \frac{y}{K}\right)y}_{f(y)}}$, $K = \frac{r}{a}$

(constant) equilibrium solutions:

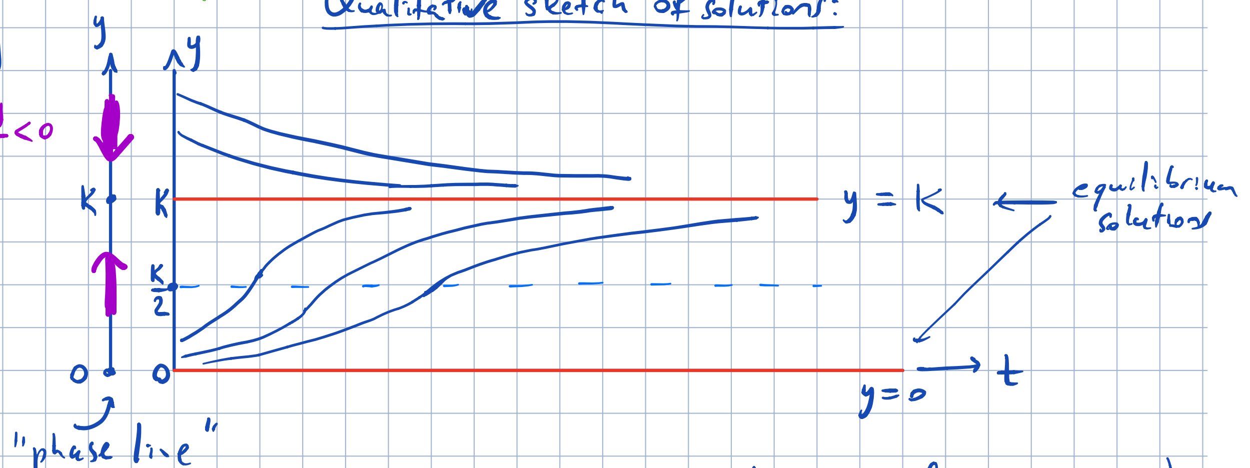
$y = 0 = \varphi_1(t)$

$y = K = \varphi_2(t)$

$\leftarrow$ zeros of $f(y)$ ("critical points")



Qualitative sketch of solutions:

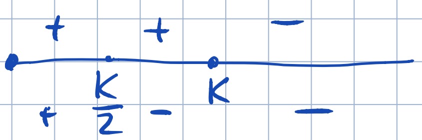


solutions approach the line $y = K$ but don't intersect it (follows from uniqueness)

Concavity of solutions: $\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt} f(y) = f'(y)\frac{dy}{dt} = f'(y) f(y)$

sol concave up if f and f' have same sign, i.e. for $0 < y < \frac{K}{2}$, $y > K$

concave down if f, f' have opposite signs, i.e. for $\frac{K}{2} < y < K$



← sign of f

← sign of f'

inflection points on a solution occur when $f'(y)=0$, i.e. when $y=\frac{K}{2}$.

K is approached but never exceeded if $y_0<K$

→ K is the "saturation level" or "environmental carrying capacity"

Note: a non-linear term in (*) created a drastically different behavior of solutions than in linear case!

Explicit solution: $\frac{dy}{\left(1-\frac{y}{K}\right)y} = r\,dt \rightsquigarrow \left(\frac{1}{y}+\frac{1/K}{1-y/K}\right)dy = r\,dt \rightsquigarrow$

$\rightarrow \ln|y| - \ln\left|1-\frac{y}{K}\right| = rt + c \rightarrow \frac{y}{1-\frac{y}{K}} = Ce^{rt} \rightarrow \boxed{y = \frac{y_0 K}{y_0+(K-y_0)e^{-rt}}}$

if $y_0=0$ then $y(t)=0$

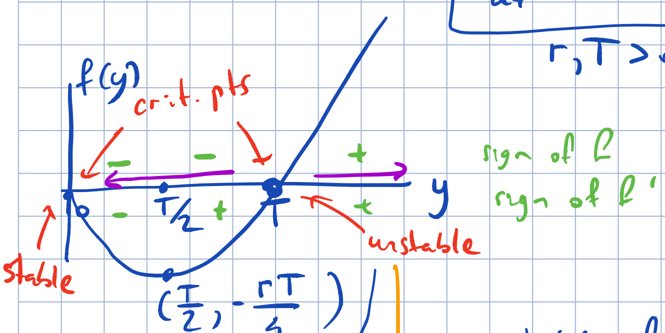
if $y_0>0$ then $\lim\limits_{t\to\infty} y(t) = K$

for each $y_0>0$, solution approaches equilibrium sol $y=K$ - asymptotically stable solution

$y=0$ - unstable equilibrium solution

(the only way to guarantee that sol remains near zero is to make sure $y_0=0$ exactly)

- Critical threshold $\boxed{\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y}$ $r, T>0$

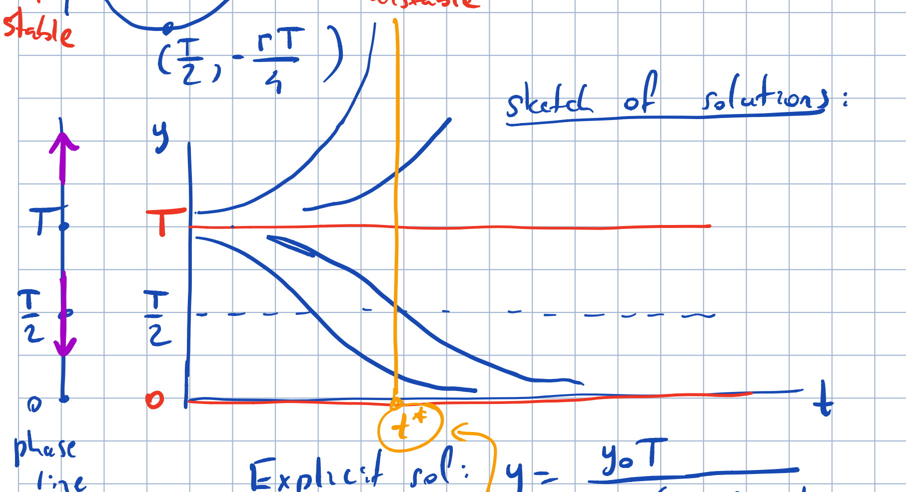


concave up if $y<\frac{T}{2}$, $y>T$

— down — $\frac{T}{2}<y<T$

inflection pt: $y=\frac{T}{2}$

sketch of solutions:



T - threshold level, below which the growth doesn't occur

for $y_0<T$, $\lim\limits_{t\to\infty} y(t) = 0$

Explicit sol: $y = \frac{y_0 T}{y_0+(T-y_0)e^{rt}}$

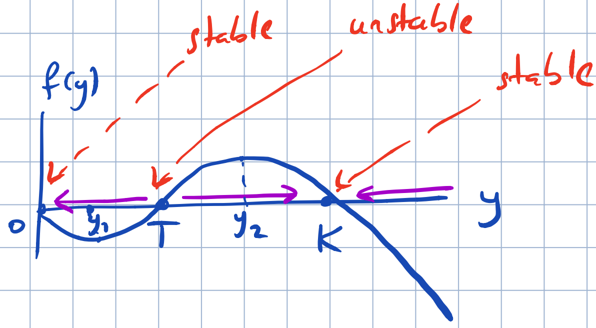
If $y_0>T$, denominator becomes zero at $t=t^* = \frac{1}{r}\ln\frac{y_0}{y_0-T}$

→ solution has a vertical asymptote at $t=t^*$

Logistic growth with a threshold

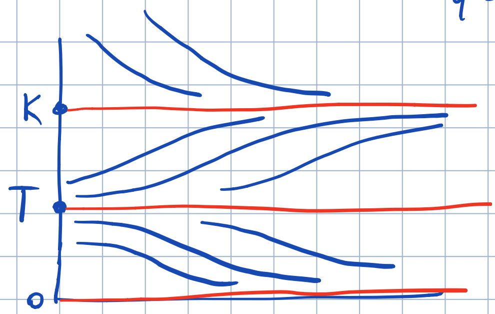
$$\frac{dy}{dt} = -r\left(1-\frac{y}{K}\right)\left(1-\frac{y}{T}\right)y$$

$r > 0$, $0 < T < K$



y_1, y_2 - roots of $f'(y) = 0$

- quadratic equation



$y = K$ stable

$y = T$ unstable

$y = 0$ stable