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## 2.6 Exact differential equations and integrating factors



**Ex** $\underbrace{2x+y^2}_{M} + \underbrace{2xy}_{N}\, y' = 0 \quad (*)$  not linear, not separable (recall separable case: $M(x)+N(y)y'=0$)

**Solution** set $\psi(x,y) = x^2 + xy^2$ note $\frac{\partial\psi}{\partial x} = 2x+y^2$, $\frac{\partial \psi}{\partial y} = 2xy \quad (**)$

Thus, $(*)$ is: $\underbrace{\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}}_{} = 0$

$= \frac{d}{dx}\psi(x, y(x))$ by chain rule.

$(*) \rightarrow \frac{d}{dx}(x^2 + x\underset{\substack{\uparrow \\ y(x)}}{y}^2) = 0 \xrightarrow[\text{integrate}]{} \psi(x,y) = \boxed{x^2+xy^2 = C}$

implicitly defines solutions of $(*)$

level curves of $\psi$ are integral curves of $(*)$

**Note**: finding $\psi$ satisfying $(**)$ was instrumental for the solution

**Generally**: $M(x,y) + N(x,y)y' = 0 \quad (\#)$

Suppose we can find $\psi(x,y)$ s.t. $\frac{\partial\psi}{\partial x} = M(x,y)$, $\frac{\partial\psi}{\partial y} = N(x,y)$

Then (if such $\psi$ exists), eq. $(\#)$ is called **exact**.

Then $(\#) \Leftrightarrow \underbrace{\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}}_{} = 0 \rightarrow$ solutions are given implicitly by $\boxed{\psi(x,y) = C}$

$\underset{\text{chain rule}}{=} \frac{d}{dx}\psi(x, y(x))$

- How to see whether such $\psi$ exists, and if so, how to find it?

**THM** Assume $M, N, \underset{\frac{\partial M}{\partial y}}{\underset{\shortparallel}{M_y}}, \underset{\frac{\partial N}{\partial x}}{\underset{\shortparallel}{N_x}}$ continuous in a rectangle $R$: $\begin{matrix}\alpha < x < \beta \\ \gamma < y < \delta\end{matrix}$

Then eq. $(\#)$ is exact (in $R$) iff $\boxed{M_y(x,y) = N_x(x,y)} \quad (@)$

I.e. there exists $\psi$ satisfying $\psi_x = M$, $\psi_y = N$ iff $M_y = N_x$

<Idea: $(@)$ is necessary for exactness: $M_y = \frac{\partial}{\partial y}\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} = N_x$ >

**How to find $\psi$**: (1) $\psi_x = M \rightarrow$ integrate in $x$ (for fixed $y$); result will contain a "constant" $h(y)$

(2) $\psi_y = N$

$\rightarrow$ substitute the result in (2) $\rightarrow$ find $h(y)$ from it

**Ex:** $\underbrace{(y\cos x + 2xe^y)}_{M} + \underbrace{(\sin x + x^2 e^y - 1)}_{N} y' = 0$

**Sol** $\left.\begin{array}{l} M_y = \cos x + 2xe^y \\ N_x = \cos x + 2xe^y \end{array}\right\}\Rightarrow$ eq. is exact!

**WANT:** $\psi(x,y)$ s.t. $\psi_x = y\cos x + 2xe^y \xrightarrow[\text{wrt } x]{\text{integrate}} \psi = y\sin x + x^2 e^y + \underline{h(y)}$

$\psi_y = \sin x + x^2 e^y - 1$ $\qquad \psi_y = \sin x + x^2 e^y + h'(y)$

So: need $h'(y) = -1 \rightarrow$ take $h = -y$

**Thus** $\psi = y\sin x + x^2 e^y - y$ and solutions are given implicitly by $\boxed{y\sin x + x^2 e^y - y = C}$

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**Ex** $\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N} y' = 0$

**Sol:** $M_y = 3x + 2y \neq N_x = 2x + y \rightsquigarrow$ eq. not exact!

still, try: $\psi_x = 3xy + y^2 \xrightarrow{\text{integrate}} \psi = \frac{3}{2}x^2 y + xy^2 + h(y) \rightarrow \psi_y = \frac{3}{2}x^2 + 2xy + h'(y)$

$\psi_y = x^2 + xy$ (@@) $\qquad \xrightarrow{(@@)} h'(y) = -\frac{1}{2}x^2 - xy$ — impossible to solve since rhs depends on $x$ and lhs does not

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**Integrating factors** **Idea** try to convert an eq. to an exact one by multiplying by $\mu(x,y)$

$M(x,y) + N(x,y)\,y' = 0 \xrightarrow[\cdot\mu(x,y)]{} \mu M + \mu N y' = 0$ ($) $\qquad$ exact iff $(\mu M)_y = (\mu N)_x$ (*)

$\Leftrightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x$

– PDE on $\mu$ – hard!

Consider the case $\mu = \mu(x)$ – indep. of $y$

$\rightarrow$ (*) becomes $\mu M_y = \mu_x N + \mu N_x$ i.e. $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$ (**)

So: if $\frac{M_y - N_x}{N}$ depends only on $x$ (not on $y$), can solve (**) for $\mu(x)$ and then solve ($) as an exact eq.

$\left(\mu_x = 0 \rightsquigarrow \mu_y = \frac{N_x - M_y}{M}\mu\right)$

**Ex:** $\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N} y' = 0$ (§) $\qquad$ Q: find $\mu$ and solve the eq.

**Sol** $\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x+y)} = \frac{1}{x}$ – indep. of $y \rightsquigarrow$ can find $\mu(x)$ from

$\frac{d\mu}{dx} = \frac{1}{x}\mu \rightarrow \mu(x) = x$ integrating factor

$(\S)\cdot\mu$ : $\underbrace{(3x^2y + xy^2)}_{\tilde{M}} + \underbrace{(x^3 + x^2y)}_{\tilde{N}} y' = 0$

$\tilde{M}_y = 3x^2 + 2xy$
$\tilde{N}_x = 3x^2 + 2xy$ $\Rightarrow$ - exact eq.! (as it should be)

$\psi_x = 3x^2y + xy^2 \xrightarrow{\text{integrate in } x} \psi = x^3y + \frac{1}{2}x^2y^2 + h(y)$

$\psi_y = x^3 + x^2y$ $\qquad$ $\psi_y = x^3 + x^2y + h'(y)$

so $h'(y) = 0$ $\rightarrow$ can take $h(y) = 0$ $\rightarrow$ $\psi(x,y) = x^3y + \frac{1}{2}x^2y^2$

Solutions are given implicitly by $\boxed{x^3y + \frac{1}{2}x^2y^2 = C}$

Remark: there is a second integrating factor $\mu(x,y) = \frac{1}{xy(2x+y)}$ - depends on $x, y$, harder to find