4/24/2020 | (3.1) Homogeneous equations with constant coefficients

2^{nd} order ODE $\quad \dfrac{d^2y}{dt^2} = f(t, y, \dfrac{dy}{dt})$

linear case: $\quad y'' + p(t)y' + q(t)y = g(t) \quad (*) \quad$ or $\quad P(t)y'' + Q(t)y' + R(t)y = G(t) \quad (**)$

Initial condition for 2^{nd} order ODE: $\quad y(t_0) = y_0, \; y'(t_0) = y_0'$
fixed numbers

Linear eq. $(*, **)$ is homogeneous if r.h.s. is zero: $g(t) = 0$, $G(t) = 0$

Today: coefficients are constants, i.e. $\boxed{ay'' + by' + cy = 0}$

Ex: $y'' - y = 0$, $y(0) = 2$, $y'(0) = -1$ — Solve IVP (init. val. prob.)

Sol: look for solutions of $y'' = y$. $\quad y_1(t) = e^t, \quad y_2(t) = e^{-t}$

also: $c_1 e^t$ and $c_2 e^{-t}$ are solutions. Note: a sum of two solutions is again a solution

Thus: $y = c_1 y_1(t) + c_2 y_2(t) = \boxed{c_1 e^t + c_2 e^{-t}}$ a solution

check: $y' = c_1 e^t - c_2 e^{-t}, \quad y'' = c_1 e^t + c_2 e^{-t} = y$ ✓

- We have a 2-parameter family of solutions. Fix c_1, c_2 from init. cond:

$$\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= c_1 - c_2 = -1 \end{aligned} \Rightarrow \begin{aligned} c_1 &= 1/2 \\ c_2 &= 3/2 \end{aligned} \Rightarrow \boxed{y(t) = \tfrac{1}{2} e^t + \tfrac{3}{2} e^{-t}} \text{ — sol of the IVP}$$

Generally: $\quad ay'' + by' + cy = 0 \quad (\#)$
real constants

Idea: try to find solutions of the form $y = e^{rt}$
to be determined

$y' = re^{rt}, \quad y'' = r^2 e^{rt}$. So, $(\#) \sim (ar^2 + br + c)e^{rt} = 0$

So, $y = e^{rt}$ a solution of $(\#)$ iff $\boxed{ar^2 + br + c = 0} \quad (@)$
"characteristic equation"

Char. eq. can have (1) two different real roots, $r_1 \neq r_2$ $\quad \leftarrow$ assume this for today
(2) two complex conjugate roots, $r_1, \; r_2 = \bar{r}_1$
(3) repeated real root, $r_1 = r_2$

$y_1(t)=e^{r_1 t}$, $y_2(t)=e^{r_2 t}$ solutions. Thus, $y(t)=C_1y_1(t)+C_2y_2(t)$

roots of char. eq.

$$=\boxed{C_1e^{r_1t}+C_2e^{r_2t}}$$ also a solution

- general solution of (#)

Given an init. cond. $y(t_0)=y_0,\ y'(t_0)=y_0'$
can determine uniquely C_1, C_2

Ex: $y''+5y'+6y=0$

(a) find gen. sol.

(b) solve the IVP $y(0)=2,\ y'(0)=3$

Sol (a): $y=e^{rt}$ a sol. iff $\underbrace{r^2+5r+6}_{(r+2)(r+3)}=0 \rightarrow r_1=-2,\ r_2=-3$ roots

So, gen. sol:
$$y=C_1e^{-2t}+C_2e^{-3t}$$

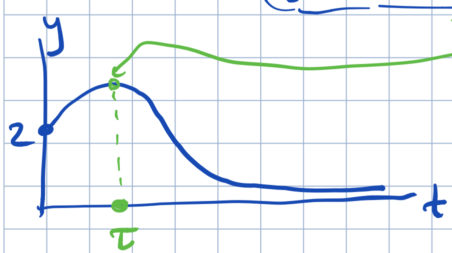
(b) $y(0)=C_1+C_2=2$

$y'(0)=-2C_1-3C_2=3$

$$\begin{bmatrix}1 & 1 & 2\\ -2 & -3 & 3\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & 7\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 9\\ 0 & 1 & -7\end{bmatrix}$$

$C_1=9$
$C_2=-7$

$$y=9e^{-2t}-7e^{-3t}$$ sol of the IVP



Q: find the location of the maximum point of the solution

Sol $y'(\tau)=0 \Rightarrow e^{\tau}=\frac{21}{18}=\frac{7}{6} \Rightarrow \tau=\ln\frac{7}{6}\approx 0.15$

$y'(\tau)=-18e^{-2\tau}+21e^{-3\tau}$

$$y(\tau)=9e^{-2\tau}-7e^{-3\tau}=9e^{-2\ln\frac{7}{6}}-7e^{-3\ln\frac{7}{6}}=9\left(\frac{7}{6}\right)^{-2}-7\left(\frac{7}{6}\right)^{-3}$$
$$=\frac{9\cdot 6^2-6^3}{7^2}=\frac{108}{49}\approx 2.2$$

thus, maximum is at $\boxed{\left(t=\ln\frac{7}{6},\ y=\frac{108}{49}\right)}$

Ex $4y''-8y'+3y=0,\ y(0)=2,\ y'(0)=\frac{1}{2}$

Sol: $4r^2-8r+3=0 \rightarrow r=\frac{8\pm\sqrt{8^2-4\cdot4\cdot3}}{8}=\frac{8\pm\sqrt{16}}{8}$ $r_1=\frac{3}{2},\ r_2=\frac{1}{2}$ - roots

$y=C_1e^{\frac{3}{2}t}+C_2e^{\frac{1}{2}t}$ - general solution

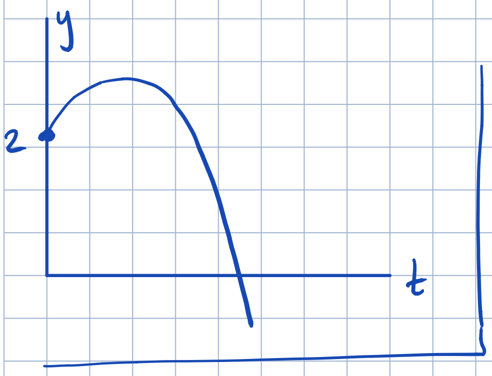
$y(0)=C_1+C_2=2$

$y'(0)=\frac{3}{2}C_1+\frac{1}{2}C_2=\frac{1}{2}$

$$\begin{bmatrix}1 & 1 & 2\\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 2\\ 3 & 1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 2\\ 0 & -2 & -5\end{bmatrix}\sim$$
$$\sim\begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{5}{2}\end{bmatrix}$$

$C_1=-1/2$
$C_2=5/2$

$$y=-\frac{1}{2}e^{3/2t}+\frac{5}{2}e^{1/2t}$$



solution of $ay''+by'+cy=0$ as $t\to\infty$

(a) $y \to 0$ if both exponents $r_{1,2}<0$

(b) $y \to \pm\infty$ if at least one exponent positive

(c) $y \to$ constant if $r_1=0$, $r_2<0$.