3.2 Solutions of linear homogeneous equations. Wronskian.

THM (Existence & uniqueness)

$$y''+p(t)y'+q(t)y=g(t)$$
$$y(t_0)=y_0,\ y'(t_0)=y_0'$$

Assume p,q,g continuous on an interval $\alpha<t<\beta$ containing t_0. Then the IVP has exactly one solution, and the solution exists for $\alpha<t<\beta$.

I.e. we have • existence, • uniqueness, • sol. exists throughout the interval where p,q,g are continuous

Ex: $y''-y=0$, $y(0)=2$, $y'(0)=-1$

we found a sol. $y=\frac{1}{2}e^{t}+\frac{3}{2}e^{-t}$ it exists for $-\infty<t<\infty$ and is unique

Ex: $(t^2-3t)y''+ty'-(t+3)y=0$; $y(1)=2$, $y'(1)=1$

Q: find the longest interval on which the sol. is certain to exist

Sol: $$y''+\underbrace{\frac{1}{t-3}}_{p(t)}y'-\underbrace{\frac{t+3}{t(t-3)}}_{q(t)}y=\underbrace{0}_{g(t)}$$

Coeffs are discontinuous at $t=0$, $t=3$

So, sol. exists for $0<t<3$

– longest interval containing t_0 where p,q,g are continuous

Note: $y''+p(t)y'+q(t)y=0$
$y(t_0)=0$, $y'(t_0)=0$ $\rightsquigarrow$ $\boxed{y=0}$ – unique solution in an interval $\alpha<t<\beta$ about t_0

THM (principle of superposition)

$y''+p(t)y'+q(t)y=0$ if y_1,y_2 are two solutions, then $\boxed{y=c_1y_1+c_2y_2}$ is also a sol. for any c_1,c_2

Indeed
$$+\begin{array}{l|l} y_1''+py_1'+qy_1=0 & \cdot c_1 \\ y_2''+py_2'+qy_2=0 & \cdot c_2 \end{array}$$
$$(c_1y_1''+c_2y_2'')+p(c_1y_1'+c_2y_2')+q(c_1y_1+c_2y_2)=0 \implies y''+py'+qy=0$$

So, starting with y_1,y_2, we construct an infinite family of solutions (*). Do we get all solutions?

– can we satisfy init. cond. $y(t_0)=y_0$, $y'(t_0)=y_0'$?

$$\begin{aligned} y(t_0)&=c_1y_1(t_0)+c_2y_2(t_0)=y_0 \\ y'(t_0)&=c_1y_1'(t_0)+c_2y_2'(t_0)=y_0' \end{aligned}$$

– can be solved for c_1,c_2 for any y_0,y_0' iff
$$\underbrace{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}_{y_1(t_0)y_2'(t_0)-y_2(t_0)y_1'(t_0)=:W}\neq 0$$

$W=W[y_1,y_2](t_0)$ – Wronskian determinant (or just Wronskian) of the solutions y_1,y_2 (at t_0)

So, if y_1, y_2 two sols of $y''+py'+qy=0$,
then one can find c_1, c_2 s.t. $y=c_1y_1+c_2y_2$ satisfies the init. cond.
$y(t_0)=y_0$, $y'(t_0)=y_0'$ for <u>any</u> y_0, y_0' iff the Wronskian $W[y_1,y_2]=y_1y_2'-y_2y_1'$ is <u>nonzero</u> at t_0.

<u>Ex</u>: $y''+5y'+6y=0$ $y_1=e^{-2t}$ $y_2=e^{-3t}$ - solutions
Wronskian: $W[y_1,y_2]=\begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t}\end{vmatrix} = -e^{-5t} \neq 0$ for all t
$\Rightarrow y_1, y_2$ can be used to construct sol. of the eq. with any init. cond. at any t_0!

<u>THM</u> Suppose y_1, y_2 are solutions of $y''+py'+qy=0$
The 2-parameter family of solutions $y=C_1y_1(t)+C_2y_2(t)$ includes <u>every</u> sol. of the eq. iff
$W[y_1,y_2](t_0)\neq 0$ for some t_0.

So, $W[y_1,y_2](t_0)\neq 0 \iff y=C_1y_1+C_2y_2$ is the <u>general solution</u>.
In this case, y_1 and y_2 are said to form a <u>fundamental set</u> of solutions

<u>Ex</u>: Suppose $y_1(t)=e^{r_1t}$, $y_2(t)=e^{r_2t}$ solutions of $y''+p(t)y'+q(t)y=0$
show that, if $r_1\neq r_2$, y_1 and y_2 form a fund. set of sols (FSS)
<u>Sol</u>: $W=\begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t}\end{vmatrix} = \underbrace{(r_2-r_1)}_{\neq 0}e^{(r_1+r_2)t}\neq 0$.

<u>Ex</u>: Show that $y_1=t^{1/2}$, $y_2=t^{-1}$ - FFS for $2t^2y''+3ty'-y=0$, $t>0$.
<u>Sol</u>: $y_1'=\frac{1}{2}t^{-1/2}$, $y_1''=-\frac{1}{4}t^{-3/2}$ $2t^2y_1''+3ty_1'-y_1=\left(2\left(-\frac{1}{4}\right)+3\cdot\frac{1}{2}-1\right)t^{1/2}=0$
$y_2'=-t^{-2}$, $y_2''=2t^{-3}$ $2t^2y_2''+3ty_2'-y_2=(2\cdot 2+3(-1)-1)t^{-1}=0$
so, y_1, y_2 are, indeed, solutions

$W[y_1,y_2]=\begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2}\end{vmatrix} = -\frac{3}{2}t^{-\frac{3}{2}}\neq 0$ for $t>0$ $\Rightarrow y_1, y_2$ - FSS $\Rightarrow \boxed{y=C_1t^{1/2}+C_2t^{-1}}$ - general sol.

• $y''+p(t)y'+q(t)y=0$
(*) let y_1 be the sol. satisfying $y(t_0)=1$, $y'(t_0)=0$
let y_2 — " — $y(t_0)=0$, $y'(t_0)=1$
Then y_1, y_2 form a FSS since $W[y_1,y_2](t_0)=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1\neq 0$

<u>Ex</u>: $y''-y=0$
$\underbrace{y_1=e^t,\ y_2=e^{-t}}$
FSS, <u>does not</u> satisfy (*) at $t_0=0$

$\underbrace{\tilde{y}_1=\frac{1}{2}e^t+\frac{1}{2}e^{-t}=\cosh t,\ \tilde{y}_2=\frac{1}{2}e^t-\frac{1}{2}e^{-t}=\sinh t}$
another FSS, satisfying (*)

- $y'' + p(t)y' + q(t)y = 0$ $\quad p, q$ - continuous real functions

  if $y = u(t) + i v(t)$ - complex-valued solution, then u, v are also solutions.

THM (Abel)

$y'' + p(t)y' + q(t)y = 0$; let p, q be continuous for $\alpha < t < \beta$ and let y_1, y_2 be solutions.

Then $W[y_1, y_2](t) = C \exp\left(-\int p(t)\,dt\right)$ (#), C - constant depending on y_1, y_2 (but not on t).

$W[y_1,y_2](t)$ is either zero for all t (if $C=0$) or else nonzero everywhere in $\alpha < t < \beta$.

Idea:

$$\begin{array}{rl} y_1'' + p y_1' + q y_1 = 0 & \big|\cdot(-y_2) \\ + \quad y_2'' + p y_2' + q y_2 = 0 & \big|\cdot y_1 \\ \hline \underbrace{y_1 y_2'' - y_2 y_1''}_{(y_1 y_2' - y_2 y_1')'} + p(y_1 y_2' - y_2 y_1') = 0 & \end{array}$$

So: $W' + p(t)W = 0$

$\to W = C e^{-\int p(t)\,dt}$

Ex: $2t^2 y'' + 3t y' - y = 0$, $t > 0$; $y_1 = t^{1/2}$, $y_2 = t^{-1}$ Verify that W is given by Abel's f-la (#)

Sol: we found $W = -\frac{3}{2} t^{-\frac{3}{2}}$

$\underbrace{y'' + \frac{3}{2t}}_{p} y' \underbrace{- \frac{1}{2t^2}}_{q} y = 0 \quad \xrightarrow[\text{Abel's formula}]{} W = C e^{-\int \frac{3}{2t} dt} = C e^{-\frac{3}{2}\ln t} = C t^{-\frac{3}{2}}$ ✓

$\left(C = -\frac{3}{2}\right)$