4/29/2020 | 3.3. Complex roots of char. eq.

$ay''+by'+cy=0$ a,b,c real $y=e^{rt}$ is a sol. iff $\underbrace{r^2+ar+b=0}_{\text{char. eq.}}$

$\underbrace{r_1,r_2 \text{ real}, r_1\neq r_2}_{\text{if } b^2-4ac>0}$ $\rightsquigarrow$ general sol. $y=C_1e^{r_1t}+C_2e^{r_2t}$

Suppose $b^2-4ac<0$ $\rightsquigarrow$ $r_1=\lambda+i\mu$, $r_2=\lambda-i\mu$ $\rightarrow$ $y_1=e^{(\lambda+i\mu)t}$, $y_2=e^{(\lambda-i\mu)t}$

• Euler's formula: $e^{it}=\cos t+i\sin t$ $\rightsquigarrow$ $\underset{e^se^{it}}{\underset{\parallel}{e^{s+it}}}=e^s(\cos t+i\sin t)$

Then $e^{z+z'}=e^ze^{z'}$ for z,z' complex

$e^{(\lambda+i\mu)t}=e^{\lambda t}(\cos\mu t+i\sin\mu t)$

One has $\frac{d}{dt}e^{rt}=re^{rt}$ complex

Ex: $y''+y'+9.25y=0$
$y(0)=2$, $y'(0)=8$

Sol: $r^2+r+9.25=0$ $r=\frac{-1\pm\sqrt{1-37}}{2}$ $r_1=-\frac{1}{2}+3i$, $r_2=-\frac{1}{2}-3i$

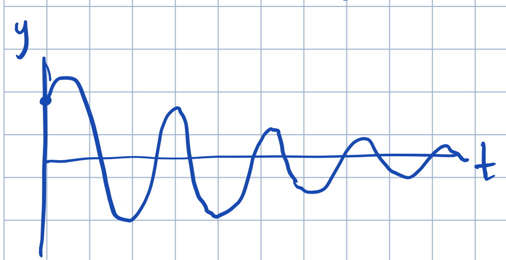
$y_1=e^{(-\frac{1}{2}+3i)t}=e^{-\frac{t}{2}}(\cos 3t+i\sin 3t)$
$y_2=e^{(-\frac{1}{2}-3i)t}=e^{-\frac{t}{2}}(\cos 3t-i\sin 3t)$
$\xrightarrow{\text{real solutions}}$
$u(t)=\operatorname{Re}y_1=e^{-t/2}\cos 3t$
$v(t)=\operatorname{Im}y_1=e^{-t/2}\sin 3t$

Wronskian: $W[u,v]=\begin{vmatrix} e^{-\frac{t}{2}}\cos 3t & e^{-\frac{t}{2}}\sin 3t \\ e^{-\frac{t}{2}}(-\frac{1}{2}\cos 3t-3\sin 3t) & e^{-\frac{t}{2}}(-\frac{1}{2}\sin 3t+3\cos 3t)\end{vmatrix}=3e^{-t}\neq 0$

So, u,v - FSS

$y=c_1e^{-\frac{t}{2}}\cos 3t+c_2e^{-\frac{t}{2}}\sin 3t$ - general solution
$=e^{-\frac{t}{2}}(c_1\cos 3t+c_2\sin 3t)$

init. cond.: $y(0)=c_1=2$, $y'(0)=-\frac{1}{2}c_1+3c_2=8$ $\rightarrow$ $c_1=2$, $c_2=3$ $\rightarrow$ $y=e^{-\frac{t}{2}}(2\cos 3t+3\sin 3t)$
- sol. of the IVP



- the solution oscillates with period $2\pi/3$ with decaying amplitude

Generally: $ay'' + by' + cy = 0$, $r = \lambda \pm i\mu$ $(\mu \neq 0)$ $\rightarrow$ $y_1 = e^{(\lambda + i\mu)t}$, $y_2 = e^{(\lambda - i\mu)t}$ complex solutions (FSS) $\rightarrow$ $u(t) = e^{\lambda t}\cos\mu t$, $v(t) = e^{\lambda t}\sin\mu t$ real FSS

$W[u,v](t) = \mu e^{2\lambda t} \neq 0$

Gen. sol.: $y = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$

Ex: $16y'' - 8y' + 155y$ $\quad y(0) = -2$, $y'(0) = 1$

Sol: char. eq. $16r^2 - 8r + 145 = 0 \rightarrow r = \dfrac{8 \pm \sqrt{64(1-155)}}{32} = \dfrac{1 \pm 12i}{4}$

$r_1 = \frac{1}{4} + 3i$, $r_2 = \frac{1}{4} - 3i$

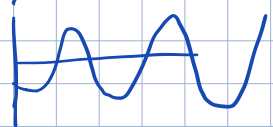
Gen. sol: $y = e^{\frac{t}{4}}(C_1 \cos 3t + C_2 \sin 3t)$

$$\left.\begin{aligned} y(0) &= C_1 = -2 \\ y'(0) &= \tfrac{1}{4}C_1 + 3C_2 = 1 \end{aligned}\right\} \rightarrow \begin{aligned} C_1 &= -2 \\ C_2 &= \tfrac{1}{2} \end{aligned}$$

$\rightarrow y = e^{\frac{t}{4}}(-2\cos 3t + \frac{1}{2}\sin 3t)$

- growing oscillation

(since the exponent is positive)

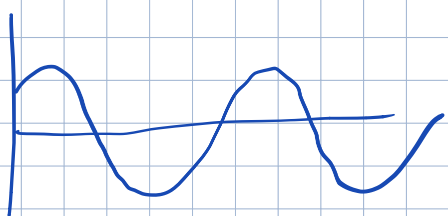


Ex: $y'' + 9y = 0$ Find gen. sol.

Sol: $r^2 + 9 = 0 \rightarrow r = \pm 3i \rightarrow$ gen. sol: $y = C_1 \cos 3t + C_2 \sin 3t$

pure oscillations of constant amplitude, period $2\pi/3$

(amplitude and phase shift determined by init. cond.)



LAST TIME: • if y_1, y_2 two sols of $y'' + p(t)y' + q(t)y = 0$

then, if <u>Wronskian</u> $W[y_1,y_2](t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$ for some t_0

the gen. sol is $y = C_1 y_1 + C_2 y_2$, <u>y_1, y_2 - FSS</u>.

• Abel's thm: $W[y_1,y_2] = C e^{-\int p(t)\,dt}$
