4/6/2020 | 2.1 Integrating factors

LAST TIME: $\frac{dy}{dt}+\frac{1}{2}y=\frac{1}{2}e^{t/3} \xrightarrow{} \frac{d}{dt}(e^{\frac{t}{2}}y)=\frac{1}{2}e^{5t/6}$

$\cdot\underbrace{\mu(t)}_{e^{t/2}}$ - integrating factor

$\to e^{\frac{t}{2}}y=\frac{3}{5}e^{\frac{5t}{6}}+C$

$\to y=\frac{3}{5}e^{t/3}+Ce^{-t/2}$

general solution

General case

$$\frac{dy}{dt}+p(t)y=g(t) \qquad (*)$$

$$\xrightarrow{\cdot\mu(t)} \underbrace{\mu(t)\frac{dy}{dt}+p(t)\mu(t)y}_{} = \mu(t)g(t) \qquad (**)$$

yet undetermined integrating factor

$$= \frac{d}{dt}(\mu(t)y) \quad \text{if} \quad \frac{d\mu(t)}{dt}=p(t)\mu(t) \rightsquigarrow$$

$$\to \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}=p(t) \xrightarrow[\text{integrate}]{} \ln|\mu(t)|=\int p(t)\,dt+k$$

$$\to \mu(t)=Ce^{\int p(t)\,dt} \xrightarrow[\substack{\text{choose}\\ C=1\\ \cdot\text{simplest solution}}]{} \boxed{\mu(t)=e^{\int p(t)\,dt}} \leftarrow \text{integrating factor}$$

- sol of

Thus, $(**)$ becomes

$$\frac{d}{dt}(\mu(t)y)=\mu(t)g(t) \longrightarrow \mu(t)y=\int\mu(t)g(t)\,dt+c$$

$$\to \boxed{y=\frac{1}{\mu(t)}\left(\int_{t_0}^{t}\mu(s)g(s)\,ds+c\right)}$$

- general sol. of $(*)$

Note: solution involves two integrations, (for μ and for y)

Ex: $ty'+2y=4t^2$

$y(1)=2$ init. condition

$\frac{1}{t}\cdot$

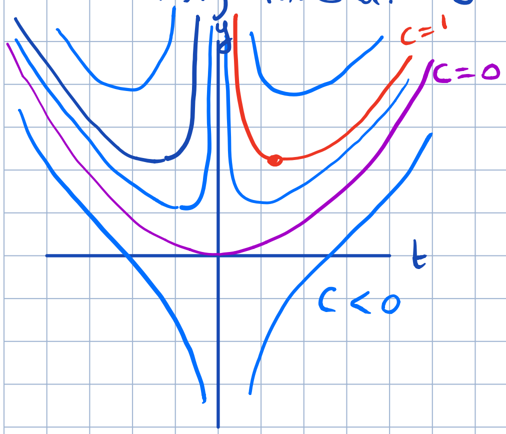
Sol: $y'+\underbrace{\frac{2}{t}}_{p(t)}y=\underbrace{4t}_{g(t)} \rightsquigarrow \mu(t)=e^{\int\frac{2}{t}dt}=e^{2\ln|t|}=t^2$

$\cdot\mu(t)$

$$\underbrace{t^2y'+2ty}_{(t^2y)'}=4t^3 \xrightarrow[\text{integrate}]{} t^2y=t^4+C \xrightarrow[\text{for } t>0]{\text{solve for } y} \boxed{y=t^2+\frac{C}{t^2}}$$

(2)

to satisfy init. cond.: $y(1) = 1 + c = 2 \Rightarrow c = 1 \Rightarrow \boxed{y = t^2 + \frac{1}{t^2}}, t > 0$



- sol. of the init. val. problem

Note: Solutions become unbounded at $t \to 0$ (due to discontinuity of $p(t)$ at $t \to 0$)

$y = t^2 + \frac{1}{t^2}$, $t < 0$ - part of general solution of (#) but not part of the solution of the init. value problem

2.2 Separable diff. eq.

First order ODE $\frac{dy}{dt} = f(t,y)$ is separable if it can be written as

$\underbrace{M(t)}_{\text{depends only on } t} + \underbrace{N(y)}_{\text{depends only on } y} \frac{dy}{dt} = 0$ or in differential form $M(t)\,dt + N(y)\,dy = 0$

- solved by integrating M and N.

Ex: $\frac{dy}{dt} = \frac{t^2}{1-y^2}$ (@) rewrite: $\underbrace{\boxed{-t^2}}_{M(t)} + \underbrace{\boxed{(1-y^2)}}_{N(y)} \frac{dy}{dt} = 0$ - separable

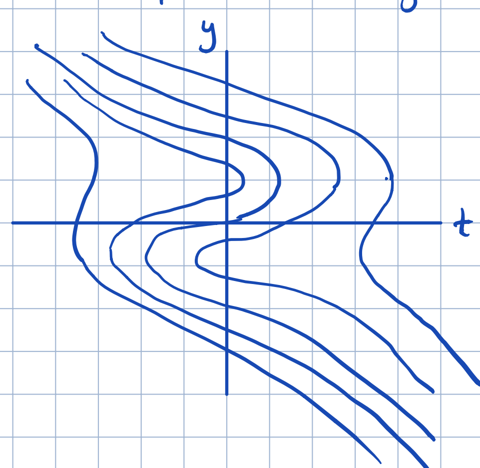
recall the chain rule:

$\frac{d}{dt} f(y) = \frac{d}{dy} f(y) \cdot \frac{dy}{dt} = f'(y) \frac{dy}{dt} \leadsto \frac{d}{dt}\left(y - \frac{y^3}{3}\right) = (1-y^2)\frac{dy}{dt}$

So, (@) becomes $\frac{d}{dt}\left(-\frac{t^3}{3} + y - \frac{y^3}{3}\right) = 0 \iff \boxed{-t^3 + 3y - y^3 = c}$ (@@)

equation for integral curves

any differentiable $\varphi(t)$ satisfying (@@) is a solution of (@)



Generally, if H_1, H_2 - anti-derivatives for M, N: $H_1'(t) = M(t)$, $H_2'(y) = N(y)$,

eq. $M(t) + N(y) \frac{dy}{dt} = 0$ becomes

$H_1'(t) + \underbrace{H_2'(y) \frac{dy}{dt}}_{\frac{d}{dt} H_2(y)} = 0 \to \frac{d}{dt}\left(H_1(t) + H_2(y)\right) = 0$

$\to \boxed{H_1(t) + H_2(y) = c}$ implicit description of solutions

if init. cond. $y(t_0) = y_0$ is given, find c from $H_1(t_0) + H_2(y_0) = c$

Ex: $\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)}$, $y(0)=-1$ - init. val. prob.

Q: find the sol., determine the interval on which the sol. exists.

Sol: $2(y-1)\,dy = (3t^2+4t+2)\,dt$

integrate $\rightsquigarrow$ $y^2-2y = t^3+2t^2+2t+c$ to satisfy init. cond. $y(0)=-1$:

$c=(-1)^2-2(-1)=3$

$\Rightarrow y^2-2y = t^3+2t^2+2t+3 \Rightarrow$ quadratic formula $y = 1 \pm \sqrt{1+(t^3+2t^2+2t+3)} =$

only $-$ satisfies init. cond.!

$= 1-\sqrt{\underbrace{t^3+2t^2+2t+4}_{(t+2)(t^2+2)}}$

Solution exists for $\underline{t>-2}$

at $t=-2$ expression under $\sqrt{--}$ vanishes