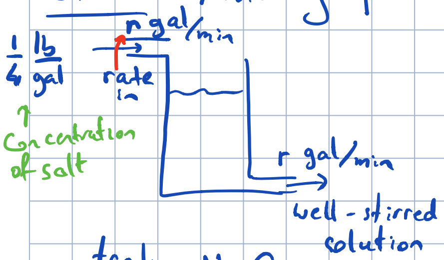
4/8/2020

2.3 Modeling with 1st order diff eq.

Ex 1 Mixing process



tank with $Q(t)$ lb of salt dissolved in 100 gal of water.

At $t=0$, $Q(t_0)=Q_0$

model:

$$\frac{dQ}{dt} = \underbrace{\text{rate (of salt) in}}_{\frac{1}{4}\cdot r} - \underbrace{\text{rate out}}_{r\cdot\frac{Q(t)}{100}}$$

← conservation of matter law

$\frac{dQ}{dt}$: rate of change of the amount of salt in the tank

$\frac{Q(t)}{100}$: concentration of salt in the tank

$$\boxed{\frac{dQ}{dt} = \frac{1}{4}r - \frac{rQ}{100}} \quad (*) \text{ diff. eq.}$$

$$\boxed{Q(0)=Q_0} \quad \text{- init. cond.}$$

physical expectation: at $t\to\infty$, concentration in the tank tends to $\frac{Q_L}{100} = \frac{1}{4}$ lb/gal $\Rightarrow \underbrace{Q_L = 25\text{ lb}}$

can find it from setting r.h.s. of (*) to zero

Analytical solution

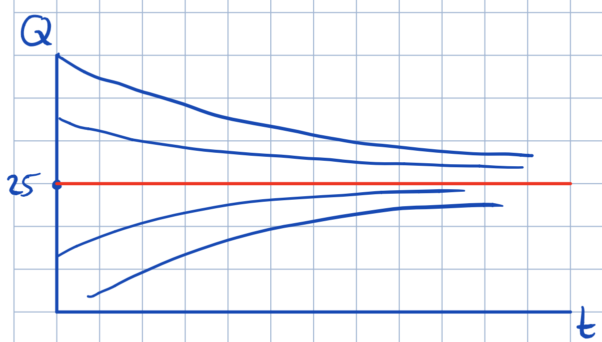
$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{1}{4}r \qquad \text{linear, 1st order}$$

integrating factor $\mu(t) = e^{\frac{r}{100}t}$. General sol.:

$$Q(t) = \frac{1}{\mu(t)}\left(\int \frac{1}{4}r\,\mu(t)\,dt + C\right) = e^{-\frac{r}{100}t}\left(25e^{\frac{r}{100}t}+C\right) = 25 + Ce^{-\frac{r}{100}t}$$

to satisfy init. cond.: $Q(0) = 25 + C = Q_0 \Rightarrow C = Q_0 - 25 \Rightarrow Q(t) = 25 + (Q_0-25)e^{-\frac{r}{100}t}$

So: $Q(t) \to 25 = Q_L$ as $t\to\infty$ - confirms the physical prediction!



Question: let $r=3$, $Q_0 = 2Q_L$

find after what time T, salt level will be within 2% of Q_L.

Sol: $Q(t) = 25(1+e^{-0.03t})$

$Q(T) = \underbrace{25\cdot(1+0.02)}_{25+2\%\text{ of }25} \Rightarrow e^{-0.03T} = 0.02 = \frac{1}{50}$

$$\Rightarrow T = \frac{\ln 50}{0.03} \approx 130.4 \text{ min}$$

Question: what should be the value of r so that $T = 45$ min?

Sol: $\frac{r\,\overset{45\text{ min}}{T}}{100} = \ln 50 \Rightarrow r = \frac{100\cdot\ln 50}{45} \approx 8.69$ gal/min

Ex 3 Chemicals in a pond

water with chemicals $\longrightarrow$ pond $\longrightarrow$ mixed water

5 mln gal/yr ; 10^7 gal of fresh (initially) water ; 5 mln gal/yr

concentration of chemicals

$\gamma(t) = 2 + \sin 2t$ g/gal - varies periodically

- construct a model
- determine the amount of chemical at any time in the pond.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 5\cdot 10^6 (2+\sin 2t) - \frac{Q}{10^7}\, 5\cdot 10^6$$

let $q(t) = \frac{Q(t)}{10^6}$. Then $\boxed{\frac{dq}{dt} + \frac{q}{2} = 10 + 5\sin 2t}$, $q(0) = 0$ (initially, water was fresh)

Integrating Factor: $\mu(t) = e^{t/2}$

$$q(t) = e^{-\frac{t}{2}}\left(\int e^{t/2}(10+5\sin 2t)\,dt + C\right)$$

$$= e^{-\frac{t}{2}}\left(20e^{t/2} + \frac{5}{\frac{1}{4}+4} e^{t/2}\left(\tfrac{1}{2}\sin 2t - 2\cos 2t\right) + C\right)$$

$$= 20 + \frac{10}{17}\sin 2t - \frac{40}{17}\cos 2t + Ce^{-t/2}$$

Aside

$$\int e^{at}\sin bt\,dt = $$

$$= \frac{1}{a}e^{at}\sin bt - \frac{b}{a}\int e^{at}\cos bt\,dt$$

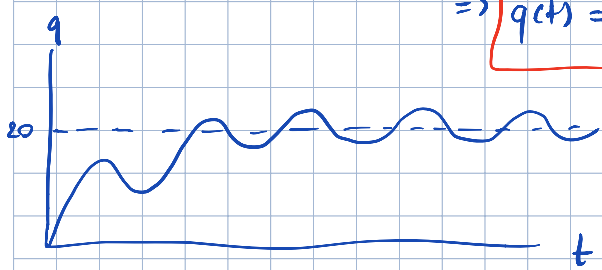
$$= \frac{1}{a}e^{at}\sin bt - \frac{b}{a^2}e^{at}\cos bt - \frac{b^2}{a^2}\int e^{at}\sin bt\,dt$$

$$\Rightarrow \int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$$

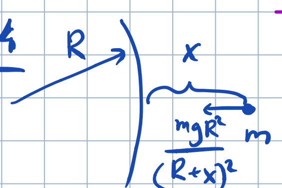
to satisfy init. cond. $q(0) = 0$:

$$20 - \frac{40}{17} + C = 0 \Rightarrow C = -\frac{300}{17}$$

$$\Rightarrow \boxed{q(t) = 20 + \frac{10}{17}\sin 2t - \frac{40}{17}\cos 2t - \frac{300}{17}e^{-t/2}}$$



Ex 4



Object is projected from Earth, perp to the surface, with initial velocity v_0. Assuming no air resistance, but taking into account the variation of Earth's gravitational field with distance,

(a) find velocity during motion

(b) find v_0 necessary to reach altitude A_{max}

(c) find smallest v_0 for which the object will not return to Earth (escape velocity)

grav. pull: $w(x) = -\frac{k}{(R+x)^2}$ ← some constant

$w(0) = -mg$ (at surface level) $\Rightarrow k = mgR^2$, $w(x) = -\frac{mgR^2}{(R+x)^2}$

So: $\begin{cases} m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2} \quad (*) \\ v(0) = v_0 \end{cases}$ ← too many variables (t, x, v)!

Remedy: let x be the indep. variable, instead of t!

Then: $\frac{dv}{dt} \underset{\text{chain rule}}{=} \frac{dv}{dx}\,\frac{dx}{dt} = v\,\frac{dv}{dx}$, $(*)$ becomes $\boxed{v\,\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}}$ – separable!

Gen. sol.: $\boxed{\frac{v^2}{2} = \frac{gR^2}{R+x} + C}$

to satisfy init. cond. $v(0) = v_0$,

$$C = \frac{v_0^2}{2} - gR$$

a) $\Rightarrow$ sol. of init. val. prob.: $v(x) = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$ – velocity as a fun. of altitude

rising falling

b) $v(A_{max}) = 0 \Rightarrow v_0^2 - 2gR + \frac{2gR^2}{R+A_{max}} = 0 \Rightarrow v_0 = \sqrt{\frac{2gRA_{max}}{R+A_{max}}}$

c) taking $A_{max} \to \infty$, we get $v_{escape} = \sqrt{2gR} \simeq 11.1\ \text{km}/\text{s}$