1/10/2022

Systems of linear equations (Poole 2.1, 2.2)

Linear equation:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

variables (indeterminates): $x_1, x_2, \dots, x_n$

coefficients: $a_1, a_2, \dots, a_n$

constant term: b

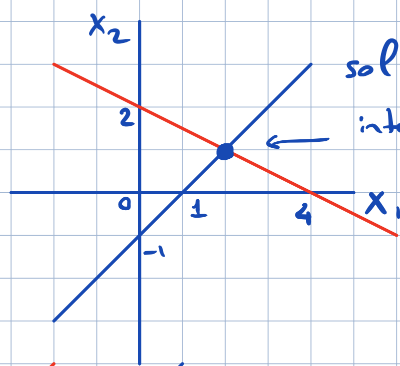
given real/complex numbers

System of linear equations:

m equations on n variables $\longrightarrow$ set of all solutions

Ex: (a) $x_1 - x_2 = 1$ (eq_1)

$x_1 + 2x_2 = 4$ (eq_2)



sol of eq_1

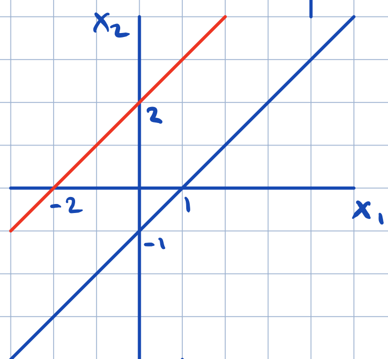
intersection - solution of the system

$(x_1, x_2) = (2, 1)$ - unique solution

sol of eq_2

(b) $x_1 - x_2 = 1$

$-2x_1 + 2x_2 = 4$



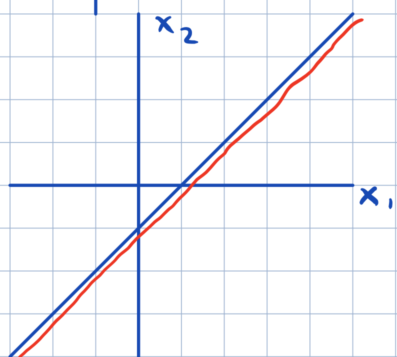
parallel lines

$\rightarrow$ no solutions

$\Rightarrow$ system is "inconsistent"

(c) $x_1 - x_2 = 1$

$-2x_1 + 2x_2 = -2$



lines coincide

$\rightarrow$ infinitely many solutions.

$x_1 = x_2 + 1$, x_2 any number

Any linear system has either

1. no solutions — system inconsistent
2. or exactly one solution
3. or infinitely many solutions

(2 and 3: system consistent)

Solving a linear system

Ex 1:

$$\underbrace{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 5x_3 &= -5 \end{aligned}}_{\text{system in a triangular (echelon) form}}$$

$\rightsquigarrow$ solve for x_3: $x_3 = -1$ $\rightsquigarrow$ solve for x_2: $x_2 = 1$ $\rightsquigarrow$ solve for x_1: $x_1 = 3$

$\Rightarrow$ the unique solution is $(3, 1, -1)$.

This method of solving a system in triangular form is called "back substitution"

Ex 2: solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 2x_1 \quad\quad + 3x_3 &= 3 \end{aligned}$$

(coefficients of each variable aligned in columns)

Matrix notation

matrix of coefficients $\longrightarrow$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 2 & 0 & 3 \end{bmatrix}$

3 rows (3 equations), 3 colums (3 variables) $\Big\} \Rightarrow$ matrix of size 3×3

augmented matrix $\begin{bmatrix} 1 & -2 & 1 & \vdots & 0 \\ 0 & 3 & -3 & \vdots & 6 \\ 2 & 0 & 3 & \vdots & 3 \end{bmatrix}$ a 3×4 matrix

added a column of right-hand sides

How to solve the system?

Idea: · use x_1 term in eq. to eliminate x_1 from other eqs

· use x_2 term in eq_2 to eliminate x_2 from other eqs, etc.

$\rightsquigarrow$ obtain a very simple equivalent linear sys. (with same solution set)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 2x_1 \quad\quad + 3x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 2 & 0 & 3 & 3 \end{bmatrix}$$

· keep x_1 in eq_1 and eliminate it from other eqs: add $(-2)eq_1$ to eq_3

$-2\cdot eq_1$	$-2x_1 + 4x_2 - 2x_3 = 0$
eq_3	$2x_1 \quad + 3x_3 = 3$
new eq_3	$4x_2 + x_3 = 3$

new system:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 4x_2 + x_3 &= 3 \end{aligned}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 0 & 4 & 1 & 3 \end{bmatrix}$$

- multiply eq_2 by $\frac{1}{3}$, to get 1 as coeff of x_2 in eq_2 (optional - simplifies the next step)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 4x_2 + x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 \cdot \frac{1}{3}$

- use x_2 term in eq_2 to eliminate x_2 from eq_3: $eq_3 \longrightarrow eq_3 - 4eq_2$

$R_3 \to R_3 - 4R_2$

$-4\,eq_2$	$-4x_2 + 4x_3 = -8$	
eq_3	$4x_2 + x_3 = 3$	
new eq_3	$5x_3 = -5$	

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 5x_3 &= -5 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

system in triangular form from Ex 1 - can solve by back substitution

- This method of solving a lin. sys. is called "Gaussian elimination"

Another way to proceed:

- $eq_3 \to \frac{1}{5} eq_3$

$R_3 \to R_3 \cdot \frac{1}{5}$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- eliminate x_3 from eq_1, eq_2:
  $eq_1 \to eq_1 - eq_3$
  $eq_2 \to eq_2 + eq_3$

$R_1 \to R_1 - R_3$
$R_2 \to R_2 + R_3$

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- eliminate x_2 from eq_1:
  $eq_1 \to eq_1 + 2eq_2$

$R_1 \to R_1 + 2R_2$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- we proved that the only sol. of the original system is $(3, 1, -1)$.
  - This method is called Gauss-Jordan elimination.

Check: (substitute into orig. sys.)

$$\begin{aligned} 1\cdot 3 - 2\cdot 1 + 1(-1) &= 0 \\ 3\cdot 1 - 3\cdot(-1) &= 6 \\ 2\cdot 3 \qquad + 3(-1) &= 3 \end{aligned}$$

✓

Solving a lin. sys. we use the operations

① replace an eq. with itself plus a multiple of another eq.

② interchange two eqs

③ multiply an eq. by a nonzero constant

for the aug. mat. we perform the corresponding elementary row operations

① (Replacement): $R_i \to R_i + cR_j$, $j \neq i$

② (Interchange): $R_i \leftrightarrow R_j$

③ (Scaling): $R_i \to cR_i$, $c \neq 0$

$\boxed{\textbf{def} \text{ Two matrices are \underline{row equivalent} iff they can be transformed one into the other by a sequence of elem. row operations}}$

- Row operations are reversible
- If the aug. mat. of two lin. systems are row equivalent, then the two systems have same sol. set.

Row reduction and echelon forms

- leading entry in a row = leftmost nonzero entry
- a rectangular matrix is in "row echelon form" (REF) if
  - all nonzero rows are above zero rows
  - each leading entry is to the left of any leading entries below it
  - ~~all entries in a column below a leading entry are zero.~~ (this is automatic)

Ex:

pivot columns

pivot positions

$$\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bullet & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet \end{bmatrix}$$

$\bullet \neq 0$ are leading entries

$*$ any entries

A matrix is in reduced row echelon form (RREF) if additionally

- all leading entries are 1
- each leading 1 is the only nonzero entry in its column.

Ex:

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$