Ex:

a matrix in REF

$$\begin{bmatrix} -2 & 0 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot positions

pivot columns

a matrix in RREF

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Any matrix can be row reduced (transformed by a sequence of elem. row operations) into more than one matrix in REF. However, $\boxed{\text{RREF of a matrix is unique.}}$

- Leading entries are always in the same position for any REF of A = pivot positions. A column containing a pivot pos. = "pivot columns"
- Leading entries in a REF of A = "pivots" (numbers)

Row reduction algorithm

$$\text{matrix } A \xrightarrow[\text{"forward phase"}]{\text{steps I-IV}} \text{REF of } A \xrightarrow[\text{"backward phase"}]{\text{step V}} \text{RREF of } A$$

Ex: $A = \begin{bmatrix} 0 & 2 & -6 & -1 & -2 \\ 2 & 1 & 9 & 9 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix}$

Step I: begin with leftmost nonzero column. It is a pivot column. Pivot position is at the top.

Step II: select a nonzero entry in pivot col. as pivot.
If necessary, interchange rows to move this entry into pivot pos.

interchange $R_1 \leftrightarrow R_3$
$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 2 & 1 & 9 & 9 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

Step III: Use row replacement to create zeros in all positions below pivot.

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

Step IV: Cover (or ignore) the row containing the pivot pos. and all rows above it
Apply steps I - III to remaining submatrix.
Repeat until there are no rows left to modify.

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{2}{3} R_2} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 \to \frac{-1}{3} R_2]{\text{(optional)}} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

new pivot

REF

new pivot

already in REF
=> Step IV stops

If we want RREF:

Step V: Beginning with rightmost pivot and working upward and to the left create zeros above each pivot. If pivot is not 1, make it 1 by rescaling rows.

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 6R_3]{R_2 \to R_2 + R_3} \begin{bmatrix} 2 & 4 & 0 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 4R_2} \begin{bmatrix} 2 & 0 & 12 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

created zeros above pivots

$$\xrightarrow[R_1 \to \frac{1}{2} R_1]{\text{rescale}} \begin{bmatrix} 1 & 0 & 6 & 0 & -6 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \text{RREF of } A.$$

Theorem (i) RREF of any matrix is unique (but REF is not)

(ii) For any REF of a matrix A, leading entries are always in the same positions - "pivot positions"

Gaussian Elimination

- algorithm for solving linear systems:

① Write down the augmented matrix of the system

② Reduce the augm. mat. to REF

③ Solve the lin. sys. corresponding to REF by back substitution.

Ex: Solve the system

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - x_2 - 2x_3 &= -5 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{5} R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

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$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$ - aug. mat. for

$x_1 - x_2 - 2x_3 = -5$
$x_2 + x_3 = 3$
$x_3 = 2$

REF

back substitution: $x_3 = 2$, $x_2 = 3 - x_3 = 1$, $x_1 = -5 + x_2 + 2x_3 = -5 + 1 + 2\cdot 2 = 0$

Solution: $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ (we will be writing solutions as column-vectors)

Ex*: solve the system

$w - x - y + 2z = 1$
$2w - 2x - y + 3z = 3$
$-w + x - y = -3$

$$\xrightarrow{\text{Aug. Mat.}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right] \xrightarrow{R_3 + 2R_2} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} w & x & y & z \end{array} \end{array}$$ REF

- aug. mat. for $w - x - y + 2z = 1$
$y - z = 1$

- infinitely many solutions

variables w, y correspond to leading entries in REF, they are "leading variables"

other observables x, z are called "free variables" - parameters for solutions

back substitution: $y = 1 + z$
$w = 1 + x + y - 2z = 2 + x - z$

- we express leading vars in terms of free vars.

assign parameters $x = s$, $z = t$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

- parametric solution of the system

Ex: Solve
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 - 2x_3 &= 1 \end{aligned} \quad \underset{\text{Aug Mat}}{\leadsto} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\to \cdots \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right] \qquad \begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_2 - x_3 &= -2 \\ 0 &= 5 \end{aligned}$$

system is inconsistent!

- **Generally:** a system is inconsistent iff REF of Aug. Mat. contains a row $[0 \cdots 0 \mid b]$, $b \neq 0$

Gauss-Jordan elimination

1. Write the aug. mat. for the lin. sys.
2. Reduce it to RREF by elem. row operations
3. If the resulting system is consistent, solve for leading variables in terms of free variables.

Ex* (revisited)

REF of Aug. Mat. $\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1+R_2]{} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} w & \underset{s}{\overset{x}{\shortparallel}} & y & \underset{t}{\overset{z}{\shortparallel}} \end{array} \end{array}$

$$\begin{aligned} w - x + z &= 2 \\ y - z &= 1 \end{aligned} \quad \to \quad \begin{aligned} w &= 2 + x - z \\ y &= 1 + z \end{aligned}$$

$$\Rightarrow \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix}$$

Def For A a matrix, <u>rank</u> of A is the number of pivots in a REF of A (= number of nonzero rows in a REF of A)
(= # pivot columns)

<u>Rank theorem</u>: Let A be the <u>coefficient matrix</u> of a lin. sys. with n variables.
If the system is consistent, then $\boxed{\#\text{ free variables} = n - \operatorname{rank} A}$

- A consistent system has: infinitely many solutions iff there are ≥ 1 free variables.

  unique sol. iff there are no free vars.

Vector equations

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} 3y \\ 4y \end{bmatrix} = \begin{bmatrix} x+3y \\ 2x+4y \end{bmatrix}$$ - linear combination of $\vec{u}, \vec{v}$

Generally, for $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ vectors in $\mathbb{R}^n$,

$x_1 \vec{v}_1 + \dots + x_k \vec{v}_k$ - linear combination

$\text{Span}(\vec{v}_1, \dots, \vec{v}_k)$ = set of all linear combinations $x_1\vec{v}_1 + \dots + x_k \vec{v}_k$ with $x_1, \dots, x_k \in \mathbb{R}$

Ex: $\vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ in $\text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right)$?

Sol: want to solve the vector eq. $x \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{u}} + y \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \quad \begin{aligned} x + 3y &= -1 \\ 2x + 4y &= 0 \end{aligned}$$

Aug. Mat.: $\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 2 & 4 & 0 \end{array}\right]$ with columns $\vec{u}$, $\vec{v}$, $\vec{w}$ $\rightarrow \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & -2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{aligned} x &= 2 \\ y &= -1 \end{aligned}$ solution

$\Rightarrow \quad \vec{w} = 2\vec{u} - \vec{v}$ - linear combination $\quad \Rightarrow \vec{w} \in \text{span}(\vec{u}, \vec{v})$

Ex: is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right)$?

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right]$$

$x_1 = -3$
$x_2 = 2$
$\boxed{0 = 2}$!

no solutions, so, NO!