# Vector equations

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} 3y \\ 4y \end{bmatrix} = \begin{bmatrix} x+3y \\ 2x+4y \end{bmatrix}$$ - linear combination of $\vec{u}, \vec{v}$

Generally, for $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ vectors in $\mathbb{R}^n$,

$x_1\vec{v}_1 + \ldots + x_k\vec{v}_k$ - linear combination

$\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_k)$ = set of all linear combinations $x_1\vec{v}_1 + \ldots + x_k\vec{v}_k$ with $x_1, \ldots, x_k \in \mathbb{R}$

Ex: $\vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ in $\mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$ ?

Sol: want to solve the vector eq. $x \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{u}} + y \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \quad \begin{aligned} x + 3y &= -1 \\ 2x + 4y &= 0 \end{aligned}$$

Aug. Mat.: $\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 2 & 4 & 0 \\ \vec{u} & \vec{v} & \vec{w} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{aligned} x &= 2 \\ y &= -1 \end{aligned}$ solution

$\Rightarrow \quad \vec{w} = 2\vec{u} - \vec{v}$ - linear combination $\quad \Rightarrow \vec{w} \in \mathrm{span}(\vec{u}, \vec{v})$

Ex: is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$?

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right]$$

$x_1 = -3$
$x_2 = 2$
$0 = 2$ !

no solutions, so, NO!

## Matrix-vector product

For $A = [\vec{a}_1 \cdots \vec{a}_n]$ an $m \times n$ matrix, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ a vector in $\mathbb{R}^n$,

columns $\in \mathbb{R}^m$

the matrix-vector product $A\vec{x} := \underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}_{\text{lin. comb. of columns of } A}$ is a vector in $\mathbb{R}^m$.

Ex: $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3$

### matrix equation

$\underbrace{A}_{\substack{\text{given} \\ m\times n \text{ mat.}}} \vec{x} = \underbrace{\vec{b}}_{\substack{\text{given vector} \\ \text{in } \mathbb{R}^m}}$ $\iff$ $\underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}}_{\text{vector eq.}}$

$A = [\vec{a}_1 \cdots \vec{a}_n]$

unknown vector in $\mathbb{R}^n$ $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Ex: $\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}}_{\vec{b}} \iff x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \iff \begin{aligned} x_1 + 2x_2 &= 3 \\ x_2 &= 2 \\ x_1 + 2x_2 &= 3 \end{aligned}$

lin. system with aug. mat.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{array}\right] = [A \mid \vec{b}]$$

- can solve for $x_1, x_2$ by row reduction.

### Operations on matrices

Let $M_{mn}$ - set of matrices $m \times n$ (# rows, # col),

for $A \in M_{mn}$, let $A_{ij}$ be the entry at row $i$, column $j$ in $A$ ("$(i,j)$-entry")

- Scalar multiple:

$$3 \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

scalar

- each matrix entry gets multiplied by the scalar

entry (1,3)

- Addition: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 6 & 6 \end{bmatrix}$

for $A, B \in M_{mn}$ matrices of <u>same size</u>, $(A+B)_{ij} = A_{ij} + B_{ij}$

$$\underset{2\times 3}{\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}} + \underset{2\times 2}{\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}$$ - not defined! (matrices of different sizes)

- <u>Matrix product</u>

<u>def</u> If $A$ is an $m \times n$ matrix, $B$ is an $n \times r$ matrix, then the product $C = AB$ is an $m \times r$ matrix with $(i,j)$-entry

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{in}B_{nj} = (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B)$$

Ex:
$$\underset{\substack{A\\ 2\times 3}}{\begin{bmatrix}1 & 0 & 5\\ 0 & 2 & -1\end{bmatrix}} \underset{\substack{B\\ 3\times 2}}{\begin{bmatrix}-1 & -1\\ 1 & 0\\ 2 & 1\end{bmatrix}} = \begin{bmatrix}1(-1)+0\cdot 1+5\cdot 2 & 1(-1)+0\cdot 0+5\cdot 1\\ 0(-1)+2\cdot 1+(-1)2 & 0(-1)+2\cdot 0+(-1)\cdot 1\end{bmatrix}$$
$$= \begin{bmatrix}9 & 4\\ 0 & -1\end{bmatrix}$$

Ex:
$$\underset{A}{\begin{bmatrix}1 & 0 & 5\\ 0 & 2 & -1\end{bmatrix}} \begin{bmatrix}-1\\ 1\\ 2\end{bmatrix} = \begin{bmatrix}1(-1)+0\cdot 1+5\cdot 2\\ 0(-1)+2\cdot 1+(-1)\cdot 2\end{bmatrix} = (-1)\begin{bmatrix}1\\ 0\end{bmatrix} + 1\cdot\begin{bmatrix}0\\ 2\end{bmatrix} + 2\cdot\begin{bmatrix}5\\ -1\end{bmatrix}$$

- linear comb. of columns of $A$
- as in def. of $A\vec{x}$.

<u>Ex:</u>
$$\underset{2\times 3}{\begin{bmatrix}1 & 0 & 5\\ 0 & 2 & -1\end{bmatrix}} \underset{2\times 2}{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}$$ - not defined!

$3 \neq 2$

<u>Properties:</u>
- $(AB)C = A(BC)$
- $AB \neq BA$ generally!
- $\overset{m\times n}{A} I_n = I_m A = A$, $I_n = \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$ $n \times n$ "identity" matrix

- Another way to compute matrix product: $A\underbrace{B}_{[\vec{b}_1 \cdots \vec{b}_r] \text{ columns}} = [A\vec{b}_1 \cdots A\vec{b}_r]$

<u>Matrix transpose</u> for $A$ an $m \times n$ matrix, its <u>transpose</u> $A^T$ is an $n \times m$ matrix with $(A^T)_{ij} = A_{ji}$

Ex: $A = \begin{bmatrix}1 & 0 & 5\\ 0 & 2 & -1\end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix}1 & 0\\ 0 & 2\\ 5 & -1\end{bmatrix}$

## Matrix inverse

For $A$ $n\times n$ matrix, $A$ is "invertible" if there exists an $n\times n$ mat. $B$ s.t. $AB = BA = I_n$. Then $B =: A^{-1}$ is called the inverse of $A$.

- $2\times 2$ case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A$ is invertible iff $ad-bc \neq 0$

"determinant"

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad ad-bc = 2\cdot 2 - 1\cdot 3 = 1$, $\quad A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

check: $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

### Gauss-Jordan method for finding $A^{-1}$

Consider the augmented matrix $\begin{bmatrix} A \mid I_n \end{bmatrix} \xrightarrow[\text{to RREF}]{\text{row reduce}} \begin{bmatrix} I_n \mid B \end{bmatrix}$ then $A^{-1} = B$

if RREF is not of this form, then $A$ is not invertible

Ex:

$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad [A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_2 - \frac{3}{2}R_1]{} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow[R_1 - 2R_2]{}$

$\rightarrow \left[\begin{array}{cc|cc} 2 & 0 & 4 & -2 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_1]{2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$

$\underbrace{\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}}_{A^{-1}}$

# Linear transformations

- A transformation (or function, or mapping) $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule assigning to each vector $\vec{v} \in \mathbb{R}^n$ some vector $T(\vec{v}) \in \mathbb{R}^m$
- A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

  (a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

  (b) $T(\underset{\substack{\uparrow \\ \text{scalar}}}{c} \cdot \vec{v}) = c\, T(\vec{v})$

<u>Ex:</u> for $A$ an $m \times n$ matrix, $\boxed{T(\vec{v}) := A\vec{v}}$ (*) is a <u>linear</u> transf. (matrix transf. determined by $A$).

<u>Theorem</u> Any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transf. (*), with

$$\underbrace{A = \left[ T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \cdots \;\; T(\vec{e}_n) \right]}_{\text{"standard matrix" of } T}$$

where $\vec{e}_i = \begin{array}{l} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i^{th} \text{ place} \\ \\ \\ \\ \end{array} \end{array} \in \mathbb{R}^n$

$= i^{th}$ column of $I_n$.

<u>Ex:</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{v}) = 3\vec{v}$ $\Rightarrow$

$$\text{stand. matrix} = \left[ \underset{T\left(\underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\right)}{\begin{pmatrix} 3 \\ 0 \end{pmatrix}} \;\; \underset{T\left(\underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\right)}{\begin{pmatrix} 0 \\ 3 \end{pmatrix}} \right]$$