Subspaces associated with a matrix

def Let A be an $m\times n$ matrix

1. The row space of A is the subspace $\text{row}(A)$ of $\mathbb{R}^n$ spanned by rows of A
2. The column space of A is the subspace $\text{col}(A)$ of $\mathbb{R}^m$ spanned by columns of A

Remark: $\text{col}(A) = \{\text{vectors of the form } A\vec{x},\ \vec{x}\in\mathbb{R}^n\}$

Ex: $A=\begin{bmatrix}1 & -1\\ 0 & 1\\ 3 & -3\end{bmatrix}$

a) is $\vec{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ in $\text{col}(A)$?

b) is $\vec{w}=[4\ \ 5]$ in $\text{row}(A)$?

c) describe $\text{row}(A)$ and $\text{col}(A)$

Sol a) $\vec{b}\in\text{col}(A)$ iff lin. sys. $A\vec{x}=\vec{b}$ is consistent

Aug. Mat. $\left[\begin{array}{cc|c}1 & -1 & 1\\ 0 & 1 & 2\\ 3 & 3 & 3\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{array}\right]$ consistent $\Rightarrow$ $\vec{b}\in\text{col}(A)$

b) $\vec{w}\in\text{row}(A)$ iff $\left[\begin{array}{c}A\\ \hline \vec{w}\end{array}\right] \xrightarrow[\substack{\text{row reduction}\\ \text{using only operations}\\ R_i + cR_j,\ i>j}]{} \left[\begin{array}{c}A'\\ \hline \vec{0}\end{array}\right]$

some matrix

zero row

- since elem. row operations create lin. comb. of rows

$\left[\begin{array}{c}A\\ \hline \vec{w}\end{array}\right] = \left[\begin{array}{cc}1 & -1\\ 0 & 1\\ 3 & -3\\ \hline 4 & 5\end{array}\right] \xrightarrow[R_4-4R_1]{R_3-3R_1} \left[\begin{array}{cc}1 & -1\\ 0 & 1\\ 0 & 0\\ \hline 0 & 9\end{array}\right] \xrightarrow{R_4-9R_2} \left[\begin{array}{cc}1 & -1\\ 0 & 1\\ 0 & 0\\ \hline 0 & 0\end{array}\right] \Rightarrow \vec{w}\in\text{row}(A)$

c) similarly $\left[\begin{array}{cc}1 & -1\\ 0 & 1\\ 3 & -3\\ \hline x & y\end{array}\right] \rightarrow \left[\begin{array}{cc}1 & -1\\ 0 & 1\\ 0 & 0\\ \hline 0 & 0\end{array}\right] \Rightarrow$ any $[x\ \ y]$ is in $\text{row}(A)$,

so $\text{row}(A)=\mathbb{R}^2$

$\text{col}(A)$: $\left[\begin{array}{cc|c}1 & -1 & x\\ 0 & 1 & y\\ 3 & -3 & z\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & -1 & x\\ 0 & 1 & y\\ 0 & 0 & z-3x\end{array}\right]$ consistent iff $z-3x=0$

so, $\text{col}(A) = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \,\middle|\, z-3x=0\right\}$

Thm If $\overset{\text{matrix}}{A}$ is row equivalent to B then $\boxed{\text{row}(A) = \text{row}(B)}$

def Let A be an $m \times n$ matrix. The <u>null space</u> of A is the set of all solutions of the homogeneous eq. $A\vec{x} = \vec{0}$. It is denoted $\text{null}(A)$.

- $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ is $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ in $\text{null}(A)$?

Sol: $A\vec{v} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ so $\vec{v} \in \text{null}(A)$.

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def A <u>basis</u> for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that

1) spans S and
2) is linearly independent

Ex: standard unit vectors $\vec{e}_1, \dots, \vec{e}_n$ in $\mathbb{R}^n$ $\left(\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ place} \right)$

- span $\mathbb{R}^n$ $\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n \right)$
- are lin. indep.

$\Rightarrow \{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$ (the "standard basis")

Ex: $S = \text{span}\underbrace{(\vec{u}, \vec{v}, \vec{w})}_{\text{not lin. indep.}: \ \vec{w} = 2\vec{u} + \vec{v}}$ where $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$S \ni c_1\vec{u} + c_2\vec{v} + c_3\underbrace{\vec{w}}_{2\vec{u}+\vec{v}} = (c_1 + 2c_3)\vec{u} + (c_2 + c_3)\vec{v}$

so, any vector in S is a lin. comb. of just $\vec{u}, \vec{v}$

$\Rightarrow S = \text{span}\underbrace{(\vec{u}, \vec{v})}_{\text{lin. indep.}}$ $\Rightarrow \{\vec{u}, \vec{v}\}$ - basis for S.

Basis for the row space

Ex: $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (RREF)

$\text{row}(A) \underset{\uparrow}{=} \text{row}(R)$ $\longleftarrow$ spanned by nonzero rows of R; they are lin. indep (due to staircase pattern)

since A and R row equivalent

$\Rightarrow \boxed{\{[1\ 0\ 1\ 0\ -1],\ [0\ 1\ 2\ 0\ 3],\ [0\ 0\ 0\ 1\ 4]\}}$ - basis for row(A)

- For any A, a basis for row(A) is given by the nonzero rows of <u>any</u> REF of A.

Basis for the column space

<u>Thm</u> For any matrix A, a basis for col(A) is given by $\underbrace{\text{pivotal columns}}_{\text{columns that in a REF of A contain a leading entry}}$ of A.

not of REF!

Ex: $A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \\ \begin{matrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{matrix} \end{array} \longrightarrow R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (RREF)

pivotal columns

So: a basis for col(A) is $\{\vec{a}_1, \vec{a}_2, \vec{a}_4\} = \boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}}$

Basis for the null space

Ex: find a basis for $\text{null}(A)$ ← 4×5 matrix above

Sol: $\text{null}(A) = (\text{set of solutions of } A\vec{x} = \vec{0})$ – compute by Gauss-Jordan elimination

aug. mat. $[A \,|\, \vec{0}] \longrightarrow [R \,|\, \vec{0}] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$x_3 = s$, $x_5 = t$ — free variables

$x_1 + x_3 - x_5 = 0$
$x_2 + 2x_3 + 3x_5 = 0$
$x_4 + 4x_5 = 0$

$$\Rightarrow \begin{aligned} x_1 &= -x_3 + x_5 \\ x_2 &= -2x_3 - 3x_5 \\ x_4 &= -4x_5 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+t \\ -2s-3t \\ s \\ -4t \\ t \end{bmatrix} = s\underbrace{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + t\underbrace{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\vec{v}} = s\vec{u} + t\vec{v}$$

So, $\text{null}(A) = \text{span}(\underbrace{\vec{u}, \vec{v}}_{\text{lin. indep.}})$

$$\Rightarrow \{\vec{u}, \vec{v}\} = \boxed{\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}} \text{ – basis for null}(A).$$

- To find a basis for $\text{null}(A)$, solve $R\vec{x} = \vec{0}$ (RREF of A) for leading vars in terms of free vars. Write the general sol. as a lin. comb. of k (k = # free vars) vectors $\vec{v}_1, \dots, \vec{v}_k$ with parameters = free vars. These $\vec{v}_1, \dots, \vec{v}_k$ form a basis for $\text{null}(A)$.

Dimension

Basis theorem Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

def If S is a subspace of $\mathbb{R}^n$, the number of vectors in a basis for S is called the **dimension** of S. Notation: $\dim S$.

Ex: $\mathbb{R}^n$ has a basis $\{\underbrace{\vec{e}_1, \dots, \vec{e}_n}_{n}\}$ $\Rightarrow \dim \mathbb{R}^n = n$.

For A $m \times n$ matrix, $\dim \text{row}(A) = \#\text{ pivots} =: \text{rank}(A)$

$\dim \text{col}(A) = \#\text{ pivots} = \text{rank}(A)$

$\dim \text{null}(A) = \#\text{ free vars in } A\vec{x} = \vec{0} =: \text{"nullity}(A)\text{"}$

<u>Rank thm:</u> $\boxed{\text{rank}(A) + \text{nullity}(A) = n}$

<u>Ex:</u> for 4×5 matrix A above, $\dim \text{row}(A) = \underset{\text{rank}}{3} = \dim \text{col}(A)$,

$\dim \text{null}(A) = 2 \leftarrow$ nullity

$$\underset{\text{rank}}{3} + \underset{\text{nullity}}{2} = \underset{\#\text{ columns}}{5}$$