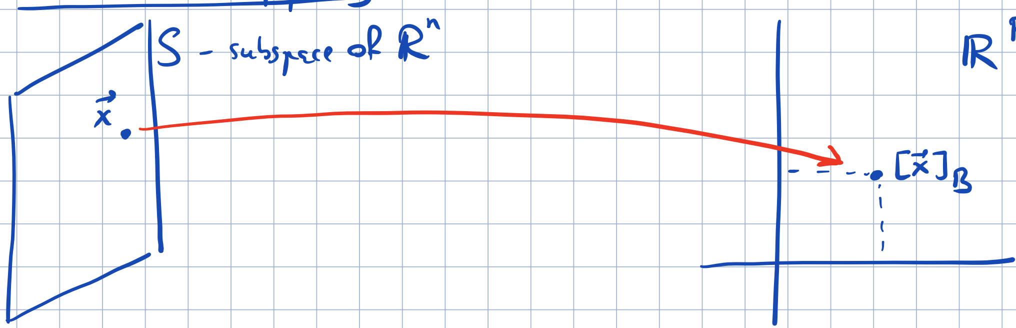
LAST TIME

- if $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ - basis for $S \subset \mathbb{R}^n$, coeffs in $\vec{v} = c_1\vec{b}_1 + \dots + c_p\vec{b}_p$ are "$\mathcal{B}$-coordinates" of $\vec{v} \in S$. $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ - $\mathcal{B}$-coordinate vector.
- if $S = \mathbb{R}^n$, then $\vec{v} = P_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}$ where $P_{\mathcal{B}} = \underbrace{[\vec{b}_1 \dots \vec{b}_n]}_{\text{invertible matrix}}$,

  $[\vec{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{v}$

Coordinate mapping



- "new name" for $\vec{x}$.

Let $\mathcal{B}$ be a basis for S. Then the coordinate mapping $[\ ]_{\mathcal{B}}: S \to \mathbb{R}^p$, $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a <u>1-1</u> lin. transf. from S <u>onto</u> $\mathbb{R}^p$.

(A linear mapping $T: S \to S'$, $S \subset \mathbb{R}^n$, $S' \subset \mathbb{R}^m$ - subspaces, which is <u>1-1</u> and <u>onto</u> is called an <u>isomorphism</u>.)

Every vector calculation in S is repeated in S' and vice versa,

E.g. $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$ in $\mathbb{R}^n$ iff $c_1[\vec{v}_1]_{\mathcal{B}} + \dots + c_k[\vec{v}_k]_{\mathcal{B}} = \vec{0}$ in $\mathbb{R}^p$

So S and S' are "the same".

Thus, a subspace S with a basis of p vectors is indistinguishable from $\mathbb{R}^p$.

Change of basis

Ex: S - subspace with two bases, $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ s.t. $\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$, $\vec{b}_2 = -6\vec{c}_1 + c_2$. Suppose that

$$\vec{x} = 3\vec{b}_1 + \vec{b}_2 \quad (*), \quad \text{i.e., } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Q: find $[\vec{x}]_{\mathcal{C}}$.

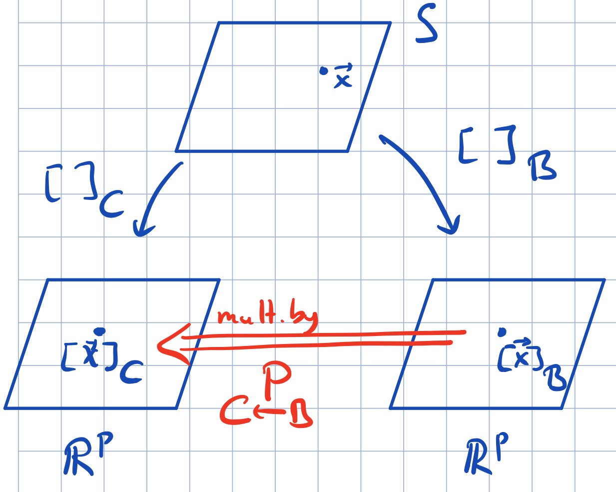
Sol: Apply the coord. mapping defined by $\mathcal{C}$ to $(*)$:

$$[\vec{x}]_{\mathcal{C}} = 3[\vec{b}_1]_{\mathcal{C}} + [\vec{b}_2]_{\mathcal{C}} = \underbrace{\big[\,[\vec{b}_1]_{\mathcal{C}}\ [\vec{b}_2]_{\mathcal{C}}\,\big]}_{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[\vec{x}]_{\mathcal{B}}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Thm Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$, $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_p\}$ be two bases for S. Then there is a unique $p \times p$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ s.t.

$$\boxed{[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\vec{x}]_{\mathcal{B}}} \quad (\#)$$

Explicitly: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \big[\,[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_p]_{\mathcal{C}}\,\big]$ — change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$



Remark: $(\#)$ implies

$$[\vec{x}]_{\mathcal{B}} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}[\vec{x}]_{\mathcal{C}}$$

Hence $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$

Change of basis in $\mathbb{R}^n$

Recall: if $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, $\mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\}$ stand. basis in $\mathbb{R}^n$, then

$$[\vec{b}_i]_{\mathcal{E}} = \vec{b}_i \quad \text{and} \quad \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$$

Change between two nonstandard bases for $\mathbb{R}^n$

Ex*: $\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$; $\vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ $\vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ two bases for $\mathbb{R}^2$

Q: find $\underset{C\leftarrow B}{P}$

Sol: we need $[\vec{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $[\vec{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

By def., $[\vec{c}_1 \ \vec{c}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}_1$, $[\vec{c}_1 \ \vec{c}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \vec{b}_2$

To solve two lin. systems simultaneously, augment the coeff. matrix by $\vec{b}_1$ and $\vec{b}_2$

$$[\vec{c}_1 \ \vec{c}_2 \mid \vec{b}_1 \ \vec{b}_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \xrightarrow[R_2+4R_1]{} \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{array}\right] \rightarrow$$

$$\xrightarrow[\frac{1}{7}R_2]{} \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{array}\right] \xrightarrow[R_1-3R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Thus $[\vec{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $[\vec{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and $\underset{C\leftarrow B}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$

Observe: $[\underbrace{\vec{c}_1 \ \vec{c}_2}_{\text{"to" basis}} \mid \underbrace{\vec{b}_1 \ \vec{b}_2}_{\text{"from" basis}}] \longrightarrow [I \mid \underset{C\leftarrow B}{P}]$ ← works analogously for any two bases for $\mathbb{R}^n$

(Gauss-Jordan method for computing $\underset{C\leftarrow B}{P}$)

Another way to construct $\underset{C\leftarrow B}{P}$:

$$\underset{C\leftarrow B}{P} = \underset{C\leftarrow \mathcal{E}}{P} \cdot \underset{\mathcal{E}\leftarrow B}{P} = (P_C)^{-1} P_B$$

or:

$$\vec{x} = P_B[\vec{x}]_B$$
$$\vec{x} = P_C[\vec{x}]_C \Rightarrow [\vec{x}]_C = P_C^{-1}\vec{x}$$
$$\Rightarrow [\vec{x}]_C = \underbrace{P_C^{-1} P_B}_{\underset{C\leftarrow B}{P}}[\vec{x}]_B$$

Ex (back to $\mathcal{B}$, $\mathcal{C}$ of Ex*)

$$\underbrace{\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}}_{\mathcal{B}} \; ; \; \underbrace{\vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \quad \vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{\mathcal{C}}$$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 42 & 28 \\ -35 & -21 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Ex: $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$ in $\mathbb{R}^3$

· find $[\vec{x}]_{\mathcal{B}}$:
$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

· find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \to$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}R_3 \\ R_2 + R_3 \\ R_1 - R_3}]{} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} & \underbrace{\boxed{\begin{matrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}}}_{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}} \end{array}\right]$$

· find $[\vec{x}]_{\mathcal{C}}$:
$$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{x}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}.$$

Rem if $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ three bases for $\mathbb{R}^n$, then
$$\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} \; \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$$
(or: $P_{\mathcal{D}}^{-1} P_{\mathcal{B}} = (P_{\mathcal{D}}^{-1} P_{\mathcal{C}})(P_{\mathcal{C}}^{-1} P_{\mathcal{B}})$)