$$\text{LAST TIME}: \quad T: \underset{\dim = n}{V} \longrightarrow \underset{\dim = m}{W}$$

$\mathcal{B} = [\vec{v}_1, \ldots, \vec{v}_n]$ $\quad C \quad$ - bases

$$\underset{W \leftarrow V}{[T]} := \big[\, [T(v_1)]_C \cdots [T(v_n)]_C \,\big]$$

- matrix of T w.r.t. bases $\mathcal{B}, C$.

$$[T(\vec{x})]_C = \underset{C \leftarrow \mathcal{B}}{[T]} \, [\vec{x}]_{\mathcal{B}}$$

Ex: $V = \operatorname{span}(\sin x, \cos x) \subset \mathcal{F}$, $\quad D : V \to V$, $\quad f(x) \mapsto f'(x)$

$[D]_{\mathcal{B}} = ?$ $\quad \mathcal{B}$ basis $\{\sin x, \cos x\}$

Sol:
$$[D]_{\mathcal{B}} = \Big[\, [\underbrace{D(\sin x)}_{\cos x}]_{\mathcal{B}} \;\; [\underbrace{D(\cos x)}_{-\sin x}]_{\mathcal{B}} \,\Big] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Ex: $D : P_3 \longrightarrow P_2$, $\quad p(x) \longmapsto p'(x)$

$\mathcal{B} = \{1, x, x^2, x^3\}$ basis for P_3

$\mathcal{C} = \{1, x, x^2\}$ basis for P_2

a) Find $\underset{C \leftarrow B}{[D]}$

Sol:

$$\underset{C\leftarrow B}{[D]} = \left[[\underbrace{1'}_{0}]_C \; [\underbrace{x'}_{1}]_C \; [\underbrace{(x^2)'}_{2x}]_C \; [\underbrace{(x^3)'}_{3x^2}]_C \right]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = A$$

b) use $\mathcal{B}' = \{x^3, x^2, x, 1\}$ for P_3 instead

$$\underset{C\leftarrow B'}{[D]} = \left[[(x^3)']_C \; [(x^2)']_C \; [(x)']_C \, , \; [1']_C \right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

c) $[D(\underbrace{5 - x + 2x^3}_{p(x)})]_C \xrightarrow{\text{can obtain directly}}$ $D(p(x)) = -1 + 6x^2$

$\Rightarrow [D(p(x))]_C = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$

can obtain as $A[p(x)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$ ✓

Ex: $T : P_2 \longrightarrow P_2$, $\quad p(x) \longmapsto p(2x-1)$ $\qquad \mathcal{B} = \{1, x, x^2\}$

$$[T]_B = \left[[\underbrace{T(1)}_{1}]_B \; [\underbrace{T(x)}_{2x-1}]_B \; [\underbrace{T(x^2)}_{4x^2-2x+1}]_B \right] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Rem Consider $I : \underset{\mathcal{B}}{V} \longrightarrow \underset{\mathcal{C}}{V}$. Then $\underset{C\leftarrow B}{[I]} = \underset{C\leftarrow B}{P}$

two bases for V

changes $[\vec{v}]_B$ to $[\vec{v}]_C$

$$\left[[\vec{v}_1]_C \cdots [\vec{v}_n]_C \right]$$

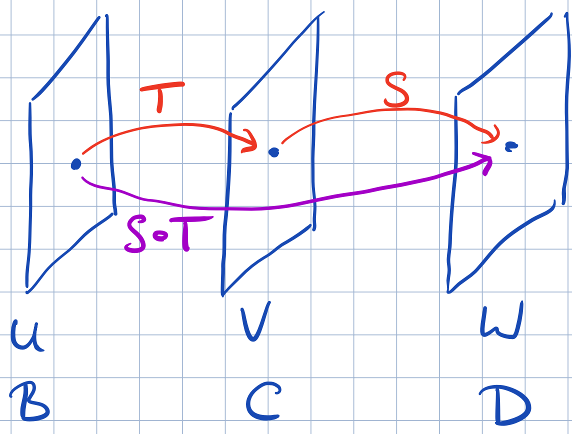
vectors of $\mathcal{B}$

Matrices of composite and inverse lin. transformations

- Let U, V, W be fin. dim. vector spaces with bases $\mathcal{B}, \mathcal{C}, \mathcal{D}$, respectively. Let $T: U \to V$ and $S: V \to W$ be lin. transformations. Then

$$\boxed{\underset{\mathcal{D} \leftarrow \mathcal{B}}{[S \circ T]} = \underset{\mathcal{D} \leftarrow \mathcal{C}}{[S]} \; \underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}}$$

"matrix of the composite transf. is the product of the matrices"



- Let $T: V \to W$ be a lin. transf. between n-dimensional v. spaces V and W. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V, W.

Then T is invertible iff the matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}$ is invertible.

In this case, $$\boxed{\underset{\mathcal{B} \leftarrow \mathcal{C}}{[T^{-1}]} = \left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}\right)^{-1}}$$

<u>Ex:</u> $T: \mathbb{R}^2 \to \mathcal{P}_1$ find T^{-1} (if T is invertible)

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + (a+b)x$$

<u>Sol:</u> Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ in $\mathbb{R}^2$, $\mathcal{E}' = \{1, x\}$ in $\mathcal{P}_1$.

$$\underset{\mathcal{E}' \leftarrow \mathcal{E}}{[T]} = \Big[\, [\underbrace{T(\vec{e}_1)}_{1+x}]_{\mathcal{E}'}, [\underbrace{T(\vec{e}_2)}_{x}]_{\mathcal{E}'} \,\Big] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \underset{\mathcal{E} \leftarrow \mathcal{E}'}{[T^{-1}]} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow [T^{-1}(a+bx)]_{\mathcal{E}} = \underset{\mathcal{E} \leftarrow \mathcal{E}'}{[T^{-1}]} \, [a+bx]_{\mathcal{E}'} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a+b \end{bmatrix}$$

Thus, $T^{-1}(a+bx) = \begin{bmatrix} a \\ b-a \end{bmatrix}$

Change of basis for a lin. transf.

Let $T: V \to V$. (same space) How are $[T]_B$ and $[T]_C$ related for B, C two bases for V?

<u>Thm</u>: Let V be a fin. dim. vector space with bases B and C and let $T: V \to V$ be a lin. transf.

Then $\boxed{[T]_C = P^{-1} [T]_B P}$ $(*)$ with $P = \underset{B \leftarrow C}{P}$

Indeed: $\underset{C \leftarrow C}{[T]} = \underbrace{\underset{C \leftarrow B}{P}}_{P^{-1}} \underset{B \leftarrow B}{[T]} \underbrace{\underset{B \leftarrow C}{P}}_{P}$ ∎

<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad B = \mathcal{E}$ stand. basis, $[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+3y \\ 2x+2y \end{bmatrix}$ $\quad$ choose $C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Then by $(*)$, $[T]_C = \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}^{-1}}_{\underset{C \leftarrow B}{P}} \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}_{[T]_B} \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}}_{\underset{B \leftarrow C}{P}} = \dots = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ - diagonal matrix!

Determinants (Poole 4.2)

Recall

- for a 2×2 matrix, $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

  another notation for determinant

- for a 1×1 matrix

  $$\det[a_{11}] = |a_{11}| = a_{11}$$

- for a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$,

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

← cumbersome formula

mnemonic rule for 3×3 matrices:

+ terms

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ & & & \uparrow & \uparrow \end{array}$$

− terms

col 1 repeated

col 2 repeated