Determinants (Poole 4.2)

Recall

- for a 2×2 matrix, $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

  another notation for determinant

- for a 1×1 matrix

$$\det[a_{11}] = |a_{11}| = a_{11}$$

- for a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$,

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

← cumbersome formula

mnemonic rule for 3×3 matrices:

+ terms

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

− terms

col 1 repeated   col 2 repeated

Note:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}}_{A_{11}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix}}_{A_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}}_{A_{13}}$$

For an $n\times n$ matrix A,

let A_{ij} be the matrix obtained from A by deleting row i and column j.
- it is an $(n-1)\times(n-1)$ matrix

$\det A_{ij}$ = "(i,j)-minor of A"

denote $C_{ij} = (-1)^{i+j}\det A_{ij}$ – "(i,j)-cofactor of A"

__def__ For $A = [a_{ij}]$ an $n \times n$ matrix, $n \geq 2$

$$\det A := \sum_{j=1}^{n} a_{1j} C_{1j} \qquad (\text{"cofactor expansion along } 1^{st} \text{ row"})$$

__Ex:__

$$\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{0} \\ 0 & 0 & 4 \\ -1 & 3 & -5 \end{vmatrix} = 1 \cdot C_{11} + 2 C_{12} + 0 \cdot C_{13} = 1 \cdot \underbrace{\begin{vmatrix} 0 & 4 \\ 3 & -5 \end{vmatrix}}_{-12} - 2 \underbrace{\begin{vmatrix} 0 & 4 \\ -1 & -5 \end{vmatrix}}_{4} + 0 \cdot \begin{vmatrix} 0 & 0 \\ -1 & 3 \end{vmatrix}$$

$$= -12 - 8 = -20$$

__Thm__ The determinant of an $n \times n$ matrix $A = [a_{ij}]$, $n \geq 2$, can be computed as

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in} \qquad (\text{cofactor expansion along row } i)$$

and also as

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj} \qquad (\text{cofactor expansion along column } j)$$

Ex:

$$\begin{vmatrix} 1 & 2 & 0 \\ \cancel{0} & \cancel{0} & \cancel{4} \\ -1 & 3 & -5 \end{vmatrix} = (-1)^{2+3} \cdot 4 \cdot \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -4 \cdot 5 = -20$$

cofactor expansion along row 2

• It is useful to do the cofactor expansion along a row/column containing many zeros

__Ex:__

$$\begin{vmatrix} \cancel{3} & -7 & 8 & 9 & -6 \\ \cancel{0} & 2 & -5 & 7 & 3 \\ \cancel{0} & 0 & 1 & 5 & 0 \\ \cancel{0} & 0 & 2 & 4 & -1 \\ \cancel{0} & 0 & 0 & -2 & 0 \end{vmatrix} = 3 \begin{vmatrix} \cancel{2} & -5 & 7 & 3 \\ \cancel{0} & 1 & 5 & 0 \\ \cancel{0} & 2 & 4 & -1 \\ \cancel{0} & 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ \cancel{0} & \cancel{-2} & \cancel{0} \end{vmatrix} = 3 \cdot 2 \cdot \overbrace{(-1)^{3+2}}^{-1} (-2) \cdot \underbrace{\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}}_{-1}$$

$$= -12$$

__Thm__ If A is a triangular matrix $\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{bmatrix}$ then $\det A$ is the product of diagonal entries of A.

Ex:

$$\begin{vmatrix} \cancel{1} & 2 & 3 \\ \cancel{0} & 4 & 5 \\ \cancel{0} & 0 & 6 \end{vmatrix} = 1 \begin{vmatrix} \cancel{4} & 5 \\ \cancel{0} & 6 \end{vmatrix} = 1 \cdot 4 \cdot |6| = 1 \cdot 4 \cdot 6$$

Properties of determinants

- if A has a row of zeros, $\det A = 0$
- if $A \xrightarrow[R_i \rightleftarrows R_j]{} B$ then $\det B = -\det A$
- if $A \xrightarrow[kR_i]{} B$ then $\det B = k \det A$
- if $A \xrightarrow[R_i + kR_j]{} B$ then $\det B = \det A$

behavior w.r.t. row operations

also works for columns

Ex:
$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \underset{\substack{R_2+2R_1 \\ R_3+R_1}}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \underset{R_2 \rightleftarrows R_3}{=} -\underbrace{\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}}_{\text{triangular}} = -1 \cdot 3 \cdot (-5) = 15$$

Ex (combining cofactor expansion and row reduction):
$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} \underset{R_4+R_2}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} \underset{\substack{\text{cofactor} \\ \text{expansion} \\ \text{along col. 1}}}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} =$$

$$\underset{R_2-3R_1}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} \underset{R_2 \rightleftarrows R_3}{=} +2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot 1 \cdot (-3) \cdot 5 = -30$$

Properties of determinants

- $\det(AB) = (\det A)(\det B)$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ $\qquad AB = \begin{bmatrix} 2 & 5 \\ 4 & 13 \end{bmatrix}$

$\det A = 3 \qquad \det B = 2 \qquad \det(AB) = 26 - 20 = 6$

$= \det A \cdot \det B$ ✓

Corollary: $\det A^{-1} = \dfrac{1}{\det A}$ for A invertible

- A is invertible iff $\det A \neq 0$

($\det A = 0 \iff$ columns of A form a lin. dep. set
$\iff$ rows of A form a lin. dep. set)

WARNING: $\det(A+B) \neq \det A + \det B$ generally

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad 1 = \det \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A+B} \neq \underbrace{\det A}_{0} + \underbrace{\det B}_{0}$

- $\det A^T = \det A$
- $\det(cA) = c^n \det A$ (not $c \det A$!)
- det is linear in i-th column (row):

$T: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear mapping:

$\vec{x} \longmapsto \det[\underbrace{\vec{a}_1 \cdots \vec{a}_{i-1}}\ \vec{x}\ \underbrace{\vec{a}_{i+1} \cdots \vec{a}_n}]$

fixed vectors in $\mathbb{R}^n$

$T(c\vec{x}) = c\,T(\vec{x})$

$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$