Eigenvectors and eigenvalues (Poole 4.1, 4.3)

def An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $\boxed{A\vec{x} = \lambda \vec{x}}$.

some scalar

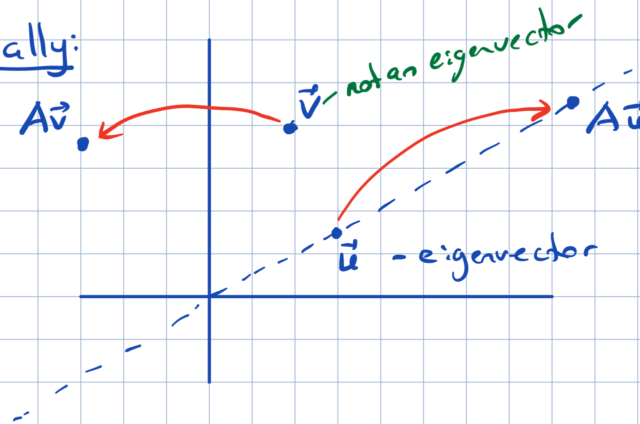
A scalar λ is called an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ has a nontriv. solution $\vec{x}$.
Such $\vec{x}$ is called an eigenvector of A corresponding to λ.

Ex:** $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\quad \vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ Q: are $\vec{u}, \vec{v}$ eigenvectors?

Sol: $A\vec{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = (-4)\vec{u} \Rightarrow \vec{u}$ is an eigenvector with $\lambda = -4$ eigenvalue

$A\vec{v} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \vec{v} \Rightarrow \vec{v}$ is not an eigenvector

Schematically:



Ex: Show that $\lambda = 7$ is an eigenvalue for $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, find the corresponding eigenvectors

Sol: $\lambda = 7$ is an eigenvalue iff $A\vec{x} = 7\vec{x}$ has a nontriv. solution

$\Leftrightarrow A\vec{x} - 7\vec{x} = \vec{0} \Leftrightarrow (A - 7I)\vec{x} = \vec{0}$

$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$ - columns are lin. dep.
$\Rightarrow$ there are nontriv. solutions of homog. eq. $\Rightarrow \lambda = 7$ is an eigenvalue

Aug. Mat.: $\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_1, $x_2 = s$ free

general sol.: $\vec{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ - each such vector with $s \neq 0$ is an eigenvector for $\lambda = 7$.

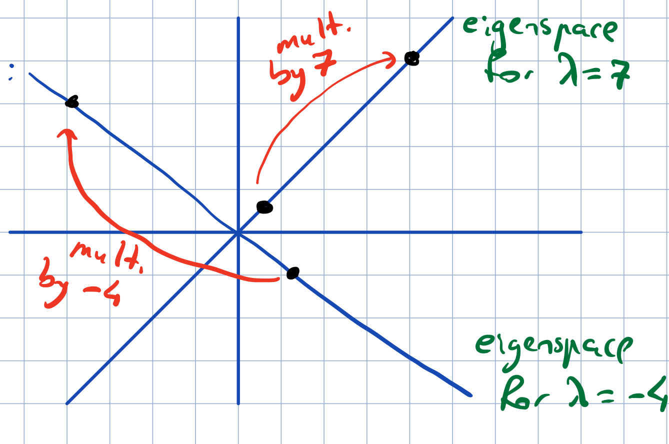
WARNING: We used row reduction of $A - \lambda I$ to find eigenvectors, but it cannot be used to find eigenvalues. REF of A has different eigenvalues than A (generally).

For A $n \times n$, λ is an eigenvalue iff $\boxed{(A - \lambda I)\vec{x} = \vec{0}}$ (*) has a nontriv. solution.

Set of solutions of (*) $= \text{null}(A - \lambda I) \subset \mathbb{R}^n$

$=$ the eigenspace E_λ of A corresponding to λ $= \{\vec{0}\} \cup \{\text{all eigenvectors for } \lambda\}$

In Ex**:



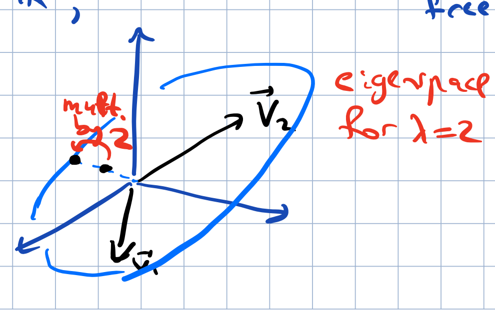
Ex: $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, $\lambda = 2$ Find the basis for the eigenspace E_2.

Sol: $A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$. Aug. Mat. for (*): $\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_1, $x_2 = s$, $x_3 = t$ free

$\vec{x} = s\underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + t\underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$. Thus, E_2 - plane in $\mathbb{R}^3$,

$\{\vec{v}_1, \vec{v}_2\}$ - basis



Thm Eigenvalues of a triangular matrix are the diagonal entries.

Idea: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$

λ - e.v. $\Leftrightarrow$ $\det(A-\lambda I) = 0$ $\Leftrightarrow$ $\lambda \in \{a_{11}, a_{22}, a_{33}\}$

$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$

Ex: $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ lower-triangular
$\lambda = 3, 0, 2$

Note: $\lambda = 0$ is an e.v. $\Leftrightarrow$ $A\vec{x} = \vec{0}$ has a nontriv. sol. $\Leftrightarrow$ A non-invertible

Invertible matrix thm (cont'd): An $n\times n$ matrix A is invertible iff:

. . . .

- iff $\det A \neq 0$
- iff 0 is not an eigenvalue of A.

Thm If $\{\vec{v}_1, \dots, \vec{v}_r\}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n\times n$ matrix A, then $\{\vec{v}_1, \dots, \vec{v}_r\}$ is a lin. indep. set.

Characteristic equation

Ex: Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol: λ e.v. $\Leftrightarrow$ $(A-\lambda I)\vec{x}$ has a nontriv. sol. $\Leftrightarrow$ $A - \lambda I$ is non-invertible

$\Leftrightarrow$ $\boxed{\det(A-\lambda I) = 0}$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21 = (\lambda+7)(\lambda-3)$$

Thus, $\det = 0$ iff $\lambda \in \{3, -7\}$. So; $\lambda = 3$, $\lambda = -7$ - eigenvalues.

- λ is an eigenvalue of A iff λ satisfies the <u>characteristic equation</u>

$$\boxed{\det(A-\lambda I)=0}$$

<u>Ex:</u> $A=\begin{bmatrix}3&1&2\\0&1&5\\0&0&3\end{bmatrix}$ Q: find the char. eq.

<u>Sol:</u> $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1&2\\0&1-\lambda&5\\0&0&3-\lambda\end{vmatrix}=(3-\lambda)^2(1-\lambda)$

So, char. eq.: $\underbrace{-(\lambda-3)^2(\lambda-1)}_{\text{char. polynomial of } A}=0$

<u>Note:</u> $\lambda=3$ - e.v. with (algebraic) multiplicity 2. (multiplicity as a root of char. eq.)

<u>Ex:</u> A 6×6, char. poly $=\lambda^6-4\lambda^5-12\lambda^4$ Q: find eigenvalues and their multiplicities

<u>Sol:</u> char. poly $=\lambda^4(\lambda-6)(\lambda+2)$. So, e.v.:

$\lambda=6$ mult. $=1$

$\lambda=0$ mult. $=4$

$\lambda=-2$ mult. $=1$

- For A $n\times n$, char. eq. has n roots (counting with multiplicities). Some of them can be <u>complex</u>.