LAST TIME $\quad$ • $A\vec{x} = \lambda \vec{x}$ — λ: eigenvalue, $\vec{x}$: eigenvector of A

①

• $E_\lambda = \text{null}(A - \lambda I)$ - eigenspace for eigenvalue λ

• eigenvalues are the roots of the characteristic equation

$$\boxed{\underbrace{\det(A - \lambda I)}_{\text{char. poly}} = 0}$$

Ex: A 6×6, char. poly $= \lambda^6 - 4\lambda^5 - 12\lambda^4$ $\quad$ Q: find eigenvalues and their multiplicities

Sol: char. poly $= \lambda^4(\lambda - 6)(\lambda + 2)$. So, e.v.:

$\lambda = 6$ $\quad$ mult. $= 1$

$\lambda = 0$ $\quad$ mult. $= 4$

$\lambda = -2$ $\quad$ mult. $= 1$

• For A $n\times n$, char. eq. has n roots (counting with multiplicities).
Some of them can be complex.

Similarity $\quad$ def $n\times n$ matrix A is "similar" to B if there is an invertible P such that

$B = P^{-1}AP$ $\quad$ (or equivalently $A = PBP^{-1}$)

Notation: $A \sim B$

• $A \longrightarrow P^{-1}AP$ - "similarity transformation"

Note: $A \sim B \Rightarrow B \sim A$

Thm If $A \sim B$, then A and B have the same char. polynomial and hence same eigenvalues (with same multiplicities)

⌈ $\det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P) = (\det P)^{-1} \cdot \det(A - \lambda I) \cdot \det P$

$= \det(A - \lambda I)$ ⌋

WARNING 1) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \not\sim \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ same e.v. and multiplicities but matrices are <u>not similar</u>.

2) Similarity $\neq$ row equivalence

↑ preserves e.v. ↑ does not preserve e.v.

Diagonalization

Often one can factorize $A = PDP^{-1}$. (D – diagonal matrix) This allows one to compute A^k efficiently for large k.

<u>Ex:</u> $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ $D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$ ... $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = PDP^{-1}$, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

then $A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$

$A^3 = PD\cancel{P^{-1}}\cancel{P}D^2P^{-1} = PD^3P^{-1}$

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$A^k = PD^kP^{-1} = \boxed{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}}$

<u>def</u> A is <u>diagonalizable</u> iff $A \sim D$ - a diagonal matrix, i.e., if $A = PDP^{-1}$ for some diagonal D and invertible P.

<u>Thm</u> ("Diagonalization thm")

An $n\times n$ matrix A is diagonalizable iff A has n <u>lin. indep.</u> eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, Then,

$\lambda_1 \dots \lambda_n$ - corresp. e.v.

$A = PDP^{-1}$ with $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$, $P = [\vec{v}_1 \cdots \vec{v}_n]$

Ex: $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ Q: diagonalize, if possible, i.e., find P, D s.t. $A = PDP^{-1}$

Sol: **Step I** Find eigenvalues of A.

char. eq. $0 = \det(A - \lambda I) = \cdots = -(\lambda-1)(\lambda+2)^2$. So, $\lambda = 1$, $\lambda = -2$ eigenvalues

Step II Find 3 lin. indep. eigenvectors. (If this fails, A cannot be diagonalized)

$\uparrow$ since A is 3×3

basis for $\lambda = 1$ eigenspace: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

basis for $\lambda = -2$ eigenspace: $\left\{ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ - lin. indep. set

Step III Construct $P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Step IV Construct $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

WARNING: the order of λ's in D should match the order of v's in P.

Check $A \overset{?}{=} PDP^{-1} \iff AP \overset{?}{=} PD$

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Q: diagonalizable?

Sol: $\lambda = 3$ is the only eigenvalue. Basis for $E_3 = \text{null}(A - 3I) = \text{null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$$\left(\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = s \\ x_2 = 0 \end{array} \Rightarrow \vec{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$x_1 = s$ free

$\Rightarrow \dim E_3 = 1 \Rightarrow$ cannot find two lin. indep. eigenvectors

$\Rightarrow A$ is not diagonalizable!

Thm An $n \times n$ matrix A with n distinct eigenvalues is diagonalizable.

Ex: $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ e.v. $\lambda = 1, 0, 2$ $\Rightarrow$ diagonalizable.

- 3 distinct values

Case of non-distinct eigenvalues

Thm Let A be $n \times n$ mat. whose distinct e.v. are $\lambda_1, \dots, \lambda_p$

$m_1, \dots, m_p$ - alg. multiplicities

(a) for each $k = 1, \dots, p$, the dimension d_k of λ_k-eigenspace is $\leq m_k$

"geometric multiplicity" of the e.v. λ_k

(b) A is diagonalizable iff $d_k = m_k$ for all k. $\left(\Leftrightarrow \sum_{k=1}^{p} d_k = n \right)$

(c) If A is diagonalizable and B_k - basis for E_{λ_k}, then

$B_1 \cup B_2 \cup \dots \cup B_p$ - basis of eigenvectors for $\mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$ Q: diagonalize if possible

Sol: $\lambda = 5, -3$ eigenvalues

$\quad\quad 2 \quad 2$ alg. multiplicities

basis for E_5: $\vec{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$ geom. mult. = 2

basis for E_{-3}: $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ geom. mult = 2

$\overset{(b)}{\Rightarrow}$ A diagonalizable

by (c), $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

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So: $A = PDP^{-1}$, $P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

Details of the two examples

$$A=\begin{bmatrix}1&3&3\\-3&-5&-3\\3&3&1\end{bmatrix} \qquad 0=\begin{vmatrix}1-\lambda&3&3\\-3&-5-\lambda&-3\\3&3&1-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&3&3\\0&-2-\lambda&-2-\lambda\\3&3&1-\lambda\end{vmatrix}$$

$$=(1-\lambda)\begin{vmatrix}-2-\lambda&-2-\lambda\\3&1-\lambda\end{vmatrix}+3\underbrace{\begin{vmatrix}3&3\\-2-\lambda&-2-\lambda\end{vmatrix}}_{0}=(1-\lambda)(-2-\lambda)\begin{vmatrix}1&1\\3&1-\lambda\end{vmatrix}=$$

$$=(1-\lambda)(-2-\lambda)^2=\boxed{-(\lambda-1)(\lambda+2)^2} \quad \Rightarrow \quad \lambda=1,\ \lambda=-2 \text{ e.v.}$$

- $\underbrace{\text{basis for } E_1}_{\text{null}(A-I)}$:

$$A-I=\begin{bmatrix}0&3&3\\-3&-6&-3\\3&3&0\end{bmatrix}\to\begin{bmatrix}1&1&0\\0&1&1\\0&0&0\end{bmatrix}\to\begin{array}{c}\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}\\ \begin{array}{ccc}x_1&x_2&x_3=s\\&&\text{free}\end{array}\end{array}$$

$x_1=s$
$x_2=-s$
$x_3=s$

$$\Rightarrow \vec{x}=s\begin{bmatrix}1\\-1\\1\end{bmatrix} \quad \Rightarrow \left\{\vec{v}_1=\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\} \text{ - basis for } E_1$$

- $\underbrace{\text{basis for } E_{-2}}_{\text{null}(A+2I)}$

$$A+2I=\begin{bmatrix}3&3&3\\-3&-3&-3\\3&3&3\end{bmatrix}\to\begin{array}{c}\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\\ \begin{array}{ccc}x_1&x_2&x_3\\&\shortparallel&\shortparallel\\&t&u\end{array}\\ \underbrace{\qquad\qquad}_{\text{free}}\end{array}$$

$x_1=-t-u$
$x_2=t$
$x_3=u$

$$\vec{x}=t\begin{bmatrix}-1\\1\\0\end{bmatrix}+u\begin{bmatrix}-1\\0\\1\end{bmatrix} \quad \Rightarrow \left\{\vec{v}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix},\ \vec{v}_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\} \text{ - basis for } E_{-2}.$$

$$A=\begin{bmatrix}5&0&0&0\\0&5&0&0\\1&4&-3&0\\-1&-2&0&-3\end{bmatrix}$$

$E_5=\text{null}(A-5I)$

$$A-5I=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&4&-8&0\\-1&-2&0&-8\end{bmatrix}\to\begin{array}{c}\begin{bmatrix}1&0&8&16\\0&1&-4&-4\\0&0&0&0\\0&0&0&0\end{bmatrix}\\ \begin{array}{cccc}x_1&x_2&x_3&x_4\\&&\shortparallel&\shortparallel\\&&s&t\end{array}\end{array}$$

$x_1=-8s-16t$
$x_2=4s+4t$
$x_3=s$
$x_4=t$

$$\Rightarrow \vec{x}=s\underbrace{\begin{bmatrix}-8\\4\\1\\0\end{bmatrix}}_{\vec{v}_1}+t\underbrace{\begin{bmatrix}-16\\4\\0\\1\end{bmatrix}}_{\vec{v}_2}$$

$\{\vec{v}_1,\vec{v}_2\}$ - basis for E_5

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$E_{-3} = \text{null}(A+3I)$

$$A+3I = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ & & \parallel & \parallel \\ & & u & w \end{array} \end{array}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = u$
$x_4 = w$

$$\vec{x} = u \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_3} + w \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_4}$$

$\{\vec{v}_3, \vec{v}_4\}$ – basis for E_{-3}.