Vector spaces (Poole 6.1)

def A vector space is a nonempty set V, objects - "vectors", with two operations:

- addition $\vec{u}+\vec{v} \in V$ for $\vec{u},\vec{v} \in V$
- multiplication by scalars $c\vec{u} \in V$ for $\vec{u} \in V$, c a scalar

such that:

1. $\vec{u}+\vec{v}$ is in V — closure under addition
2. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ — commutativity of +
3. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ — associativity of +
4. there exists an element $\vec{0}$ in V - the "zero vector" - such that $\vec{u}+\vec{0}=\vec{u}$
5. for any $\vec{u} \in V$ there exists an element $-\vec{u} \in V$ s.t. $\vec{u}+(-\vec{u}) = \vec{0}$.
6. $c\vec{u}$ is in V — closure under scalar multiplication
7. $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$
8. $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

   (7, 8) distributivity
9. $c(d\vec{u}) = (cd)\vec{u}$ — assoc of scalar product
10. $1\vec{u} = \vec{u}$

Note: this def. doesn't specify what kind of objects V consists of, and doesn't specify what the two operations look like.

For scalars in $\mathbb{R}$, V is a <u>real</u> vector space

For scalars in $\mathbb{C}$, V is a <u>complex</u> vector space

<u>Corollaries</u>: $\vec{0}$ is unique, $-\vec{u}$ is unique, $0\cdot\vec{u}=\vec{0}$, $(-1)\vec{u} = -\vec{u}$.

$(\vec{0}' = \vec{0}+\vec{0}' = \vec{0})$ $\quad ((-\vec{u})' = (-\vec{u})+\vec{u}+(-\vec{u})' = -\vec{u})$

<u>Main example up to now</u>: spaces $\mathbb{R}^n$, $n \geq 1$ with usual vector addition and scalar multiplication.

<u>Ex</u>: $V = \{\text{all } 2\times 3 \text{ matrices}\}$ with usual matrix addition and scalar multiplication.

<u>note</u>: a "vector" in V is a 2×3 matrix.

For any $m, n \geq 1$ we have $V = \{m \times n \text{ matrices}\} =: \underbrace{M_{mn}}_{\text{notation}}$
a vector space

Ex: $P_2 = \{\text{polynomials of degree} \leq 2 \text{ with real coeffs}\}$

if $p(x) = a_0 + a_1 x + a_2 x^2$, $q(x) = b_0 + b_1 x + b_2 x^2$ two elements in P_2 (polynomials)

then: $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$ - element in P_2

and $c\,p(x) = ca_0 + ca_1 x + ca_2 x^2$ - element in P_2

zero-vector in P_2: $\vec{0}$ = zero polynomial (all coeffs $= 0$) $0 + 0 \cdot x + 0 \cdot x^2$

negative: $-p(x) = -a_0 - a_1 x - a_2 x^2$

Generally: $P_n = \{\text{polynomials of degree} \leq n\}$ - vector space

$P = \{\text{all polynomials}\}$ - vector space

Ex: $V = \{\text{all real-valued functions on a set } \mathbb{D}\}$ e.g. interval $[a, b]$ in $\mathbb{R}$

addition: for $f, g \in V$, $(f+g)(x) = f(x) + g(x)$

Scalar multiplication: $(cf)(x) = c\,f(x)$

zero vector: function $f_0(x) = 0$ for any $x \in \mathbb{D}$

negative: $(-f)(x) = -f(x)$

E.g. $\mathbb{D} = \mathbb{R}$, $f(x) = 1 + \sin 3x$, $g(x) = 2 + 7x$

then: $(f+g)(x) = 3 + \sin 3x + 7x$, $(2g)(x) = 4 + 14x$

Each function is a "point" (or "vector") in V.

def A subset W of a vector space V is called a subspace of V if

a) $\vec{0} \in W$

b) $\vec{u} + \vec{v} \in W$ if $\vec{u}, \vec{v} \in W$ (W closed under addition)

c) $c\vec{u} \in W$ if $\vec{u} \in W$, $c \in \mathbb{R}$ (W closed under scalar multiplication)

operations of V

• a subspace $W \subset V$ is automatically a vector space.

Ex: $W = \{\vec{0}\} \subset V$ is a subspace ("zero subspace")

Ex $\underset{\substack{\uparrow \\ \text{all polynomials}}}{\mathbb{P}} \subset \{\text{all functions on } \mathbb{R}\}$ - subspace

$P_n \subset P$, $n \geq 0$ - subspace

Ex: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ (not even a subset!)

but $\left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$ that "look and acts" like $\mathbb{R}^2$

Subspace spanned by a set

Thm If $\vec{v}_1, \dots, \vec{v}_p$ are in V, then

(a) $W = \operatorname{span}(\vec{v}_1, \dots, \vec{v}_p)$ is a subspace of V
$:= \{c_1\vec{v}_1 + \dots + c_p\vec{v}_p\}$

(b) $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_p)$ is the smallest subspace of V containing $\vec{v}_1, \dots, \vec{v}_p$.

Terminology: W is the subspace spanned by $\{\vec{v}_1, \dots, \vec{v}_p\}$
$\{\vec{v}_1, \dots, \vec{v}_p\}$ - spanning set for W.

Ex: $W = \left\{ \text{vectors of form } \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$ show that $W \subset \mathbb{R}^4$ is a subspace

Sol: $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$. So, $W = \operatorname{span}(\vec{v}_1, \vec{v}_2)$ - subspace of $\mathbb{R}^4$.

Ex: Show that $W = \{\underbrace{a + bx - bx^2 + ax^3}_{a(1+x^3) + b(x - x^2)} \mid a, b \in \mathbb{R}\} \subset P_3$ is a subspace.

Sol: $W = \operatorname{span}(1 + x^3, x - x^2)$ - subspace

Ex: $W = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\} \subset M_{22}$ is a subspace

$$\underbrace{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \quad \Rightarrow \quad W = \mathrm{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

- We also interested in sets that span the entire V.

Ex: polynomials $1, x, x^2$ in P_2 span P_2, since each $p(x) \in P_2$
$= a_0 + a_1 x + a_2 x^2$
is a lin. comb. of $1, x, x^2$.

Likewise, $1, x, \dots, x^n$ in P_n span P_n.

Ex: $M_{22} = \mathrm{Span}\left(\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}} \right)$

since $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} E_{11} + a_{12} E_{12} + a_{21} E_{21} + a_{22} E_{22}$

$M_{mn} = \mathrm{Span} \{ E_{ij} \}_{\substack{i=1 \dots m \\ j=1 \dots n}}$ $\qquad E_{ij}$ = matrix where (i,j)-entry is 1
all other entries are 0.

Ex: $V = \mathbb{R}^2$ with operations $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{\text{new addition}}{\oplus} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + 1 \\ x_2 + y_2 \end{bmatrix}$,

$c \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c x_1 + c - 1 \\ c x_2 \end{bmatrix}$

zero-vector: $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

negative: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} -x_1 - 1 \\ -x_2 \end{bmatrix}$

– a vector space!