Dimension

Basis theorem: If a vector space V has a basis of n vectors, then every basis for V has exactly n vectors.

- A v. sp. V is called finite-dimensional if it has a basis consisting of finitely many vectors. Dimension of V ($\dim V$) is the number of vectors in a basis for V.
- $\dim\{\vec{0}\} = 0$ (convention)
- A v.sp. that has no finite basis is called infinite-dimensional.

Ex: $\mathbb{R}^n$ has a basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ $\Rightarrow \dim \mathbb{R}^n = n$

Ex: P_n has a basis $\{1, x, \dots, x^n\} \Rightarrow \dim P_n = n+1$

Ex: M_{mn} has a basis $\{E_{ij}\}_{\substack{1\le i\le m \\ 1\le j\le n}}$ $\Rightarrow \dim M_{mn} = mn$

Ex: P and F are ∞-dimensional (each contains an infinite lin. dep. set $1, x, x^2, \dots$)

Thm Let V be a v.space with $\dim V = n$. Then:

a) any lin. indep. set in V contains at most n vectors

b) any spanning set for V contains at least n vectors

c) a lin. indep. set of exactly n vectors in V is a basis for V

d) a spanning set of exactly n vectors in V is a basis for V.

e) any lin. indep. set in V can be extended to a basis for V

f) any spaning set in V can be reduced to a basis for V.

Ex: is $S = \{\overset{p_1}{1}, \overset{p_2}{1+x}, \overset{p_3}{1+x+x^2}, \overset{p_4}{x^2}\}$ a basis for P_2?

Sol: - No: Since $\underbrace{\#S}_{4} > \underbrace{\dim P_2}_{3}$, S cannot be lin. indep. - by (a)

In fact: S spans $P_2 \underset{(f)}{\Rightarrow}$ can be reduced to a basis:

$p_4 = p_3 - p_2 \Rightarrow$ can exclude p_4 and $S' = \{p_1, p_2, p_3\}$ is still spanning

$\underset{(d)}{\Rightarrow}$ S' is a basis for P_2

Ex: a) is $S = \{1-x, 1+x\}$ a basis for P_2?

- it is lin. indep. but cannot be spanning, since $\underbrace{\#S}_{2} < \underbrace{\dim P_2}_{3}$

b) extend S to a basis for P_2

idea: adjoin to S the stand basis $\qquad S' = \underbrace{\{1-x, 1+x, \cancel{1}, \cancel{x}, x^2\}}_{\text{spanning but lin. dep. (by (a))}}$

$\rightsquigarrow$ exclude lin. dep. vectors among the adjoined ones, one-by-one:

$$1 = \frac{1}{2}(1+x) + \frac{1}{2}(1-x)$$
$$x = \frac{1}{2}(1+x) - \frac{1}{2}(1-x)$$

- lin. combinations, can be excluded

$\Rightarrow \{1-x, 1+x, x^2\}$ - basis for P_2.

Thm Let W be a subspace of a fin. dim. v. space V. Then:

(a) W is finite-dimensional and $\dim W \le \dim V$

(b) $\dim W = \dim V$ iff $W = V$.

Ex: $W = \{A \in M_{22} \mid A^T = A\}$ symmetric 2×2 matrices

$$= \Big\{ \underbrace{\begin{bmatrix} a & b \\ b & d \end{bmatrix}}_{a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_{12}+E_{21}} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}} \Big\}$$

$\nearrow$ $\mathcal{B} = \{E_{11}, E_{22}, E_{12}+E_{21}\}$ - basis for $W \Rightarrow \dim W = 3$.

Linear transformations

def A <u>linear transformation</u> T from a v.sp. V to a v.sp. W is a mapping $T: V \to W$ s.t. for all $\vec{u}, \vec{v} \in V$ and all scalars c,

(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $T(c\,\vec{u}) = c\,T(\vec{u})$

- $T: V \to W$ is a lin. transf. iff

$T(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) = c_1 T(v_1) + \dots + c_k T(v_k)$ for all $\vec{v}_1, \dots, \vec{v}_k \in V$, $c_1, \dots, c_k$ scalars

Ex: $D: \mathcal{D} \to \mathcal{F}$, $D(f) = f' = \dfrac{df}{dx}$ - derivative of f

functions on $\mathbb{R}$ with continuous derivative

all functions on $\mathbb{R}$

D is a lin. transf: $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ ✓

$D(cf) = (cf)' = cf' = c\,D(f)$ ✓

Ex: $T: M_{nn} \to M_{nn}$, $T(A) = A^T$

$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$

$T(cA) = (cA)^T = cA^T = c\,T(A)$

Ex: for each V, W one has <u>zero transformation</u> $T_0: V \to W$, $\vec{v} \mapsto \vec{0}$

Ex: for each V, one has <u>identity transf.</u> $I: V \to V$, $\vec{v} \mapsto \vec{v}$

Ex: Suppose $T: \mathbb{R}^2 \to \mathcal{P}_2$, $T\Big(\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{v}_1}\Big) = 1 + x^2$, $T\Big(\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\vec{v}_2}\Big) = 1 + x$

(i) find $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$ (ii) find $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$

Sol: $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ - basis for $\mathbb{R}^2$, $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix} \Rightarrow \begin{matrix}c_1 = a\\ c_2 = b-a\end{matrix}$

(ii) $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\underbrace{(1+x^2)}_{T(\vec{v}_1)} + c_2\underbrace{(1+x)}_{T(\vec{v}_2)} = a(1+x^2) + (b-a)(1+x)$

$= b + (b-a)x + ax^2$

(i) $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = -1 - 2x + x^2$

a, b

Kernel and range (Poole 6.5)

For a lin. transf. $T: V \to W$,

$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\} \underset{\text{subspace}}{\subset} W$

$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \underset{\text{subspace}}{\subset} V$

Ex: for A $m \times n$ matrix, $T: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{v} \mapsto A\vec{v}$

$\text{range}(T) = \text{col}(A)$
$\ker(T) = \text{null}(A)$

def For $T: V \to W$ lin. transf.,

$\text{rank}(T) := \dim \text{range}(T)$
$\text{nullity}(T) := \dim \text{null}(T)$

Rank-nullity Thm: For $T: V \to W$, $\text{rank}(T) + \text{nullity}(T) = \dim V$

Ex: $D: P_3 \to P_2$, $p(x) \mapsto p'(x)$ find ker, range, rank, nullity

Sol: $\ker(D) = \{\underset{a+bx+cx^2+dx^3}{p(x)} \in P_3 \mid D(p) = 0\} = \underset{\text{constant polynomials}}{\{a\}}$ $\Rightarrow$ nullity $= \dim \ker = 1$

$\text{range}(D) = \{\underset{a+bx+cx^2}{p(x)} \in P_2 \mid p(x) = q'(x) \text{ for some } q(x) \in P_3\} = \underset{\text{entire}}{P_2}$ (i.e. D is "onto")

$= \frac{d}{dx}\underbrace{\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\right)}_{q(x)}$

$\Rightarrow$ rank $= \dim$ range $= 3$.

def $T: V \to W$ is "one-to-one" if T maps distinct vectors in V to distinct vectors in W

T is "onto" if $\text{range}(T) = W$.

Thm
- $T: V \to W$ is one-to-one iff $\ker(T) = \{\vec{0}\}$
- if $V = W$, $T: V \to W$ is 1-1 iff it is onto.
- if $T: V \to W$ is 1-1, image of a lin. indep. set in V is a lin. indep. set in W.

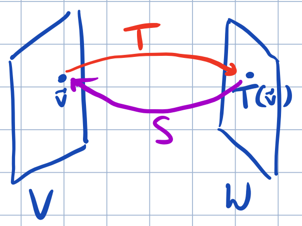
def A lin. transf. $T: V \to W$ that is 1-1 and onto is called an isomorphism.

- If $T : U \to V$, $S : V \to W$ lin. transf,

can form the composition $S \circ T : U \to W$,
$\vec{u} \longmapsto S(T(\vec{u}))$

- $T : V \to W$ is <u>invertible</u> if there exists $S : W \to V$ s.t. $\begin{cases} S \circ T = I_V \\ T \circ S = I_W \end{cases}$

Then $S =: T^{-1}$ is the <u>inverse transformation</u> of T.



- T is invertible iff it is an isomorphism.

Ex: $T : P_n \to \mathbb{R}^{n+1}$ - isomorphism
$a_0 + a_1 x + \dots + a_n x^n \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$

Ex: $T : M_{22} \to P_3$ - isomorphism
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto a + bx + cx^2 + dx^3$

- Two vector spaces (over $\mathbb{R}$) V, W are <u>isomorphic</u> iff $\dim V = \dim W$
(an isomorphism $T : V \to W$ exists)