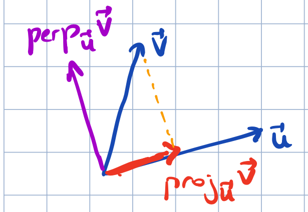
LAST TIME

Orthogonal projection:

$$\boxed{\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}}$$

orthog. projection of $\vec{v}$ onto $\vec{u}$

$\vec{v}, \vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$



$$\vec{v} = \underbrace{\text{proj}_{\vec{u}}\vec{v}}_{\parallel \vec{u}} + \underbrace{\text{perp}_{\vec{u}}\vec{v}}_{\perp \vec{u}}$$

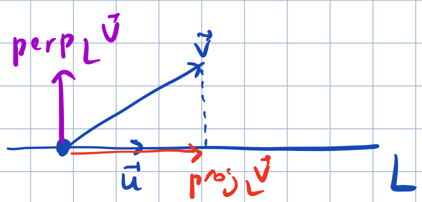
Rem $\text{proj}_{\vec{u}}\vec{v} = \text{proj}_{c\vec{u}}\vec{v}$ for any $c \neq 0$. So, it is actually a projection onto the line $L = \text{span}(\vec{u})$.

Ex: $\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Q: find $\text{proj}_{\vec{u}}\vec{v}$, $\text{perp}_{\vec{u}}\vec{v}$

Sol: $\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$; $\text{perp}_{\vec{u}}\vec{v} = \vec{v} - \text{proj}_{\vec{u}}\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Q: find the distance from $\vec{v}$ to $L = \text{span}(\vec{u})$.

Sol: $\text{dist}(\vec{v}, L) = \text{dist}(\vec{v}, \underbrace{\text{proj}_L\vec{v}}_{\text{closest point to } \vec{v} \text{ on } L}) = \|\text{perp}_L\vec{v}\| = \left\|\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\| = \sqrt{5}$



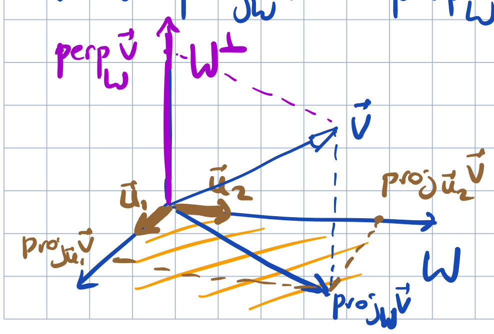
Orthogonal decomposition theorem

Let $W \subset \mathbb{R}^n$ be a subspace and $\{\vec{u}_1, \dots, \vec{u}_p\}$ an orthogonal basis for W.
Then for any $\vec{v} \in \mathbb{R}^n$ there are unique vectors $\hat{\vec{v}} \in W$, $\vec{z} \in W^{\perp}$ such that $\vec{v} = \hat{\vec{v}} + \vec{z}$. Explicitly:

$$\hat{\vec{v}} = \boxed{\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\,\vec{u}_1 + \dots + \frac{\vec{v}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\,\vec{u}_p =: \text{proj}_W\vec{v}}$$

– the orthog. projection of $\vec{v}$ onto W,

$\vec{z} = \vec{v} - \text{proj}_W\vec{v} =: \text{perp}_W\vec{v}$ – the component of $\vec{v}$ orthogonal to W.



Rem • $\text{proj}_W \vec{v} = \text{proj}_{\vec{u}_1} \vec{v} + \dots + \text{proj}_{\vec{u}_p} \vec{v}$ - sum of projections onto basis vectors

• if $\vec{v} \in W$, then $\text{proj}_W \vec{v} = \vec{v}$.

• case $W = \mathbb{R}^n$: $\vec{v} = \underbrace{\boxed{\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}} \vec{u}_1 + \dots + \boxed{\frac{\vec{v}\cdot\vec{u}_n}{\vec{u}_n\cdot\vec{u}_n}}}_{\text{coordinates of } \vec{v} \text{ w.r.t. the orthog. basis}} \vec{u}_n$

Ex: $W = \mathbb{R}^2$, $\underbrace{\vec{u}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}1\\-1\end{bmatrix}}_{\text{orthog. basis } \mathcal{B} \text{ for } \mathbb{R}^2}$, $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$

$$\vec{v} = \underbrace{\boxed{\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}}_{5/2} \vec{u}_1 + \underbrace{\boxed{\frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}}}_{-1/2} \vec{u}_2 = \frac{5}{2}\vec{u}_1 - \frac{1}{2}\vec{u}_2$$

- found the coefficients without solving the lin. system!

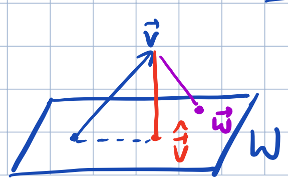
$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}5/2\\-1/2\end{bmatrix}$$

Ex*: $\underbrace{\vec{u}_1 = \begin{bmatrix}2\\5\\-1\end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix}-2\\1\\1\end{bmatrix}}_{\text{orthog. basis for } W = \text{span}(\vec{u}_1, \vec{u}_2)}$, $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ Q: find $\text{proj}_W \vec{v}$, $\text{perp}_W \vec{v}$

Sol:
$$\text{proj}_W \vec{v} = \underbrace{\boxed{\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}}_{\frac{9}{30} = \frac{3}{10}} \vec{u}_1 + \underbrace{\boxed{\frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}}}_{\frac{3}{6} = \frac{5}{10}} \vec{u}_2 = \frac{3}{10}\begin{bmatrix}2\\5\\-1\end{bmatrix} + \frac{5}{10}\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix}$$

$$\text{perp}_W \vec{v} = \vec{v} - \text{proj}_W \vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix} = \begin{bmatrix}7/5\\0\\14/5\end{bmatrix} \Rightarrow \vec{v} = \underbrace{\begin{bmatrix}-2/5\\2\\1/5\end{bmatrix}}_{\in W} + \underbrace{\begin{bmatrix}7/5\\0\\14/5\end{bmatrix}}_{\in W^\perp}$$

Thm ("Best approximation thm")
Let $W \subset \mathbb{R}^n$ subspace, $\vec{v} \in \mathbb{R}^n$, $\hat{\vec{v}} = \text{proj}_W \vec{v}$.
Then $\hat{\vec{v}}$ is the closest point on W to $\vec{v}$. I.e., for all $\vec{\omega} \in W$, $\vec{\omega} \neq \hat{\vec{v}}$, one has $\|\vec{v} - \vec{\omega}\| > \|\vec{v} - \hat{\vec{v}}\|$.



$\hat{\vec{v}}$ is the best approximation of $\vec{v}$ by an element of W

$\|\vec{v} - \hat{\vec{v}}\|$ is the "error of approximation"

Back to Ex*: $\hat{\vec{v}} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$ - closest point to $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on W.

$$\mathrm{dist}(\vec{v}, W) = \mathrm{dist}(\vec{v}, \hat{\vec{v}}) = \| \underbrace{\vec{v} - \hat{\vec{v}}}_{\mathrm{perp}_W \vec{v}} \| = \left\| \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix} \right\| = \frac{7}{5} \left\| \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\| = \frac{7}{5}\sqrt{5} = \frac{7}{\sqrt{5}}$$

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def A set of vectors in $\mathbb{R}^n$ is an orthonormal (o/n) set if it is an orthogonal set of unit vectors. An orthonormal basis for $W \subset \mathbb{R}^n$ is a basis for W which is an orthonormal set.

(I.e. $\{\vec{v}_1, \dots, \vec{v}_p\}$ is o/n iff $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$)

Ex: $\{\vec{e}_1, \dots, \vec{e}_n\}$ - o/n basis for $\mathbb{R}^n$

Ex: $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ - o/n basis for $\mathbb{R}^2$

- normalization of the orthog. set $\left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Ex: $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ is an orthogonal (not o/n) basis for $\mathbb{R}^3$. Construct an o/n basis out of it.

Sol: $\|\vec{u}_1\| = \sqrt{11}$, $\|\vec{u}_2\| = \sqrt{6}$, $\|\vec{u}_3\| = \frac{\sqrt{54}}{2} = \frac{3}{2}\sqrt{6}$

thus: $\vec{v}_1 = \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \frac{1}{3\sqrt{6}} \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$ is an $\boxed{\text{o/n}}$ basis for $\mathbb{R}^3$

Thm An $m \times n$ matrix U has o/n columns iff $U^T U = I$

Thm Let U be an $m \times n$ matrix with o/n columns; let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then:

a) $\|U\vec{x}\| = \|\vec{x}\|$ b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$ c) $U\vec{x} \perp U\vec{y}$ iff $\vec{x} \perp \vec{y}$

i.e. mapping $\begin{matrix} \mathbb{R}^n \to \mathbb{R}^m \\ \vec{x} \mapsto U\vec{x} \end{matrix}$ preserves lengths and orthogonality

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If $m = n$, square mat. U with o/n columns is called an orthogonal matrix

U is orthogonal iff $U^{-1} = U^T$.

Ex: matrix of rotation by angle φ: $U = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

$$U^{-1} = \cancel{\frac{1}{\cos^2\varphi + \sin^2\varphi}} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = U^T \quad \Rightarrow \quad U \text{ is an orthogonal matrix.}$$

Thm If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an o/n basis for $W \subset \mathbb{R}^n$, then

(1) $\mathrm{proj}_W \vec{v} = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{v} \cdot \vec{u}_p)\vec{u}_p$.

(2) If $U = [\vec{u}_1 \cdots \vec{u}_p]$ then $\boxed{\mathrm{proj}_W \vec{v} = \underbrace{U U^T}_{\text{matrix of projection onto } W.} \vec{v}}$ for all $\vec{v} \in \mathbb{R}^n$