Complex eigenvalues

Ex*: $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ Q: find eigenvalues & eigenvectors

Sol: char eq. $0 = \begin{vmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{vmatrix} = \lambda^2 - 1.6\lambda + 1$ solutions: $\lambda = \frac{1.6 \pm \sqrt{(-1.6)^2-4}}{2} = 0.8 \pm 0.6i$

for $\lambda = 0.8 - 0.6i$,

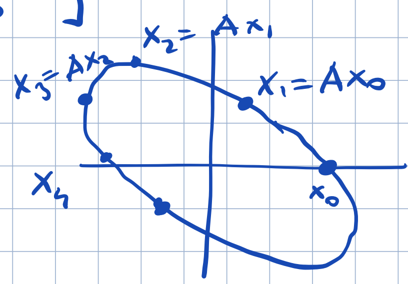
$A - \lambda I = \begin{bmatrix} -0.3+0.6i & -0.6 \\ 0.75 & 0.3+0.6i \end{bmatrix}$ (1) $(-0.3+0.6i)x_1 - 0.6x_2 = 0$
(2) $0.75x_1 + (0.3+0.6i)x_2 = 0$

nontriv. sol. exists $\Rightarrow$ both eqs determine the same relation between x_1 and x_2, (1) $\Leftrightarrow$ (2)

$\Leftrightarrow x_1 = -(0.4+0.8i)x_2$ choose $x_2 = 5 \Rightarrow$ basis for the eigenspace: $\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

similarly, for $\lambda = 0.8+0.6i$, eigenvector $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$

- mapping $\vec{x} \longmapsto A\vec{x}$ is "essentially" a rotation:



- for A a matrix with real entries,

$A\vec{x} = \lambda\vec{x} \quad \Rightarrow \quad A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$

complex conjugation $(\overline{a+ib} = a-ib)$

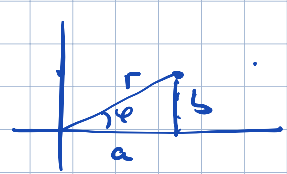
so: complex eigenvalues $\lambda = a+ib$, $b \neq 0$ occur in conjugate pairs.

In Ex*: $\lambda = 0.8 - 0.6i$ $\overline{\lambda} = 0.8 + 0.6i$ - conjugate

$\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$ - conjugate

Ex: $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with a, b real, nonzero. Eigenvalues: $\lambda = a \pm ib$ and

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\substack{\text{scaling by} \\ r = |\lambda| = \sqrt{a^2+b^2}}} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\substack{\text{rotation by} \\ \varphi = \text{argument} \\ \text{of } a+ib}}.$$



Back to Ex* $\quad A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} \quad \lambda = 0.8 - 0.6i \quad \vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

Let $P = [\operatorname{Re}\vec{v}_1 \quad \operatorname{Im}\vec{v}_1] = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$

Let $C = P^{-1} A P = \dots = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ - pure rotation by $\varphi = \arctan \frac{0.6}{0.8}$

since $|\lambda| = \sqrt{0.8^2 + 0.6^2} = 1$

Thus: $\quad A = P \underbrace{C}_{\text{rotation}} P^{-1}$

$\vec{x} = P\vec{u}$

change of variable

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, "P^{-1}", "\text{change of var.}"'] & A\vec{x} \\ \vec{u} \arrow[r, "C", "\text{rotation}"'] & C\vec{u} \arrow[u, "P"', "\text{change of var.}"] \end{tikzcd}$$

<u>Thm</u> Let A be a real 2×2 matrix with complex eigenvalue $\lambda = a - ib$, $b \neq 0$ and $\vec{v}$ the corresp. eigenvector in $\mathbb{C}^2$. Then

$$A = PCP^{-1} \quad \text{with } P = [\operatorname{Re}\vec{v} \quad \operatorname{Im}\vec{v}], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$