LAST TIME

Linear 1st order ODE

$$\frac{dy}{dx} + P(x)y = f(x) \quad \rightsquigarrow \quad \mu(x) = e^{\int P(x)\,dx}$$ integrating factor

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x) f(x)\,dx + C\right)$$ - general solution

Exact equations

- 1st order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is "exact" if

there exists $f(x,y)$ such that $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$.

Then: $\boxed{f(x,y) = C}$ - family of implicit solutions

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ - criterion of exactness

How to find f?

(1) $\frac{\partial f}{\partial x} = M(x,y) \xrightarrow[\text{in } x]{\text{integrate}} f = \int M(x,y)\,dx + \underbrace{g(y)}_{\text{"constant" of integration}}$

(2) $\frac{\partial f}{\partial y} = N(x,y) \longrightarrow$ substitute f into (2) and find g from it.

Ex: $\underbrace{2xy}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0$ check: $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ ✓ 1

$\Rightarrow$ eq. is exact.

(1) $\frac{\partial f}{\partial x} = 2xy \xrightarrow[\text{int. in } x]{} f(x,y) = x^2y + g(y)$

(2) $\frac{\partial f}{\partial y} = x^2-1 \leadsto \frac{\partial f}{\partial y} = x^2 + g'(y) = x^2-1 \Rightarrow g'(y) = -1 \Rightarrow g(y) = -y$

So: $f(x,y) = x^2y - y \Rightarrow$ implicit solution: $\boxed{x^2y - y = C}$

$\rightarrow y = \frac{C}{x^2-1}$ explicit sol. (on any interval not containing $x=1, x=-1$)

Integrating factors If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, we can try to make it exact by multiplying by an integrating factor $\mu(x,y)$:

$\boxed{\underbrace{\mu M}_{\tilde{M}}\,dx + \underbrace{\mu N}_{\tilde{N}}\,dy = 0}$ We want $\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x}$

$\frac{\partial \tilde{M}}{\partial y} = \mu M_y + \mu_y M \qquad \frac{\partial \tilde{N}}{\partial x} = \mu N_x + \mu_x N$

$\Rightarrow \boxed{\mu_x N - \mu_y M = \mu(M_y - N_x)}$ (#) - complicated PDE on μ!

Case $\mu = \mu(x)$. Then (#) is: $\mu_x N = \mu(M_y - N_x)$

$\Leftrightarrow \boxed{\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}}$ - if the rhs depends only on x, we can solve for $\mu(x)$

Similarly, case $\mu = \mu(y)$: $\frac{\mu_y}{\mu} = -\underbrace{\frac{M_y - N_x}{M}}$
if this depends on y only, can solve for $\mu(y)$.

Ex:

$$\underbrace{xy}_{M}\,dx + \underbrace{(2x^2+3y^2)}_{N}\,dy = 0 \qquad (**)$$

$M_y = x$, $N_x = 4x$ $\Rightarrow$ eq. not exact!

- $\dfrac{M_y - N_x}{N} = \dfrac{-3x}{2x^2+3y^2}$ depends on y (not on x only) $\Rightarrow$ cannot find $\mu(x)$.

- $-\dfrac{M_y - N_x}{M} = \dfrac{3x}{xy} = \dfrac{3}{y}$ depends on y only $\Rightarrow$ can find $\mu(y)$, $\dfrac{\mu_y}{\mu} = \dfrac{3}{y}$

$\rightarrow \mu = e^{3\ln y} = y^3$ $\qquad \rightarrow y^3\cdot(**)$: $\underbrace{xy^4}_{\tilde{M}}\,dx + \underbrace{(2x^2y^3+3y^5)}_{\tilde{N}}\,dy = 0$ - exact

(1) $f_x = xy^4 \xrightarrow[\text{int in } x]{} f = \dfrac{x^2y^4}{2} + g(y)$

(2) $f_y = 2x^2y^3 + 3y^5 \longrightarrow f_y = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 \rightarrow g'(y) = 3y^5$

$\rightarrow g(y) = \frac{1}{2}y^6$

$\Rightarrow f = \dfrac{x^2y^4}{2} + \dfrac{y^6}{2}$

$\Rightarrow \boxed{\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 = C}$ - implicit solution.

__Summary:__ · ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is __exact__ if $M_y = N_x$.

Integrating factor: · If $\dfrac{M_y - N_x}{N}$ depends only on x, can find $\mu(x)$ s.t. $\dfrac{\mu_x}{\mu} = \dfrac{M_y - N_x}{N}$

· If $-\dfrac{M_y - N_x}{M}$ depends only on y, can find $\mu(y)$ s.t. $\dfrac{\mu_y}{\mu} = -\dfrac{M_y - N_x}{M}$

IVP for n-th order linear ODE:

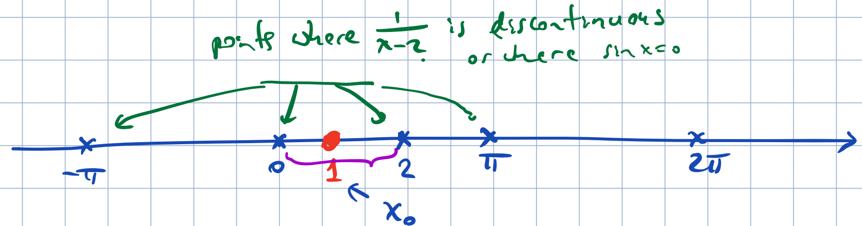
$$\begin{cases} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ y(x_0)=y_0,\ y'(x_0)=y_1, \dots, y^{(n-1)}(x_0)=y_{n-1} \end{cases}$$

Thm (Existence and uniqueness for linear ODEs)

Let $a_n(x), \dots, a_0(x)$, $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every $x \in I$. If $x_0 \in I$, then a solution $y(x)$ of the IVP exists on I and is unique.

Ex: $\begin{cases} 2y'' + 3y' + 4y = 0 \\ y(0)=0,\ y'(0)=0 \end{cases}$ — solution exists and is unique on $I=(-\infty,\infty)$ ($y=0$ is a sol. $\Rightarrow$ it is the only solution)

Ex: $\begin{cases} \sin(x)\,y'' + \dfrac{1}{x-2}\,y = 1 \\ y(1)=0,\ y'(1)=1 \end{cases}$



So, sol. exists and is unique for $x \in (0,2)$.

linear ODEs	non-linear ODEs
possible points of discontinuity of the solution can be identified by finding points of discontinuity of coefficients $a_1 \dots a_n, g$ (and zeros of a_0)	$\leftarrow$ cannot be! (and can depend on the init. condition)
- can find the interval of existence without solving the eq. explicitly.	

Ex: $\begin{cases} y' = y^2 \\ y(0)=1 \end{cases} \rightsquigarrow \dfrac{dy}{y^2} = dx \Rightarrow -y^{-1} = x + C \Rightarrow y = \dfrac{-1}{x+C}$

$\rightsquigarrow 1 = \dfrac{-1}{0+C} \Rightarrow C=-1 \Rightarrow y = \dfrac{1}{1-x}$ solution

interval of existence $I = (-\infty, 1)$

point $x=1$ is not remarkable in any way from the eq!

in fact: $\begin{cases} y' = y^2 \\ y(0)=y_0 \end{cases} \Rightarrow y = \dfrac{y_0}{1-y_0 x}$

interval of existence: $I = (-\infty, \frac{1}{y_0})$ if $y_0 > 0$,

$I = (\frac{1}{y_0}, \infty)$ if $y_0 < 0$.

More examples

Ex: $\frac{dy}{dx} = -\frac{3xy+y^2}{x^2+xy} \longrightarrow \underbrace{(3xy+y^2)}_{M}\,dx + \underbrace{(x^2+xy)}_{N}\,dy = 0 \quad (*)$

$M_y = 3x+2y$
$N_x = 2x+y$ $\Big\} \not= \;\rightarrow$ eq. is not exact!

try to find an integrating factor μ: $\frac{M_y - N_x}{N} = \frac{x+y}{x^2+xy} = \frac{1}{x}$ - depends on x only,

so, we can find $\mu(x)$ from $\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N} = \frac{1}{x} \Rightarrow \frac{d\mu}{\mu} = \frac{dx}{x} \Rightarrow \ln\mu = \ln x + C_1$

$\Rightarrow$ can take $\mu = x$

$(*)\cdot\mu$: $\underbrace{(3x^2y+xy^2)}_{\tilde{M}}\,dx + \underbrace{(x^3+x^2y)}_{\tilde{N}}\,dy = 0 \quad (**)$

check: $\tilde{M}_y = 3x^2+2xy$
$\tilde{N}_x = 3x^2+2xy$ $\Big\} \Rightarrow (**)$ exact

solve: (1) $f_x = 3x^2y+xy^2 \xrightarrow[\text{int in } x]{} f = x^3y + \frac{x^2}{2}y^2 + g(y)$

(2) $f_y = x^3+x^2y \longrightarrow f_y = x^3 + x^2y + g'(y) = x^3 + x^2y \Rightarrow g'(y) = 0$

$\Rightarrow g = C_1$

can take $g = 0$

$\Rightarrow f = x^3y + \frac{x^2y^2}{2}$

$\Rightarrow$ implicit solutions of $(*)$: $\boxed{x^3y + \frac{x^2y^2}{2} = C}$