Dot product, length, orthogonality

def For $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, the <u>dot product</u> is
(also, "inner product")

$$\boxed{\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n} \quad \text{- number}$$
$$= \vec{u}^T \vec{v}$$

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ $\vec{u}\cdot\vec{v} = 1\cdot 2 + 2\cdot 0 + 3(-1) = -1$

Properties: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

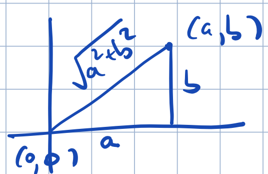
$$\left.\begin{aligned} (c\vec{u})\cdot\vec{v} &= c(\vec{u}\cdot\vec{v}) \\ (\vec{u}_1+\vec{u}_2)\cdot\vec{v} &= \vec{u}_1\cdot\vec{v} + \vec{u}_2\cdot\vec{v} \end{aligned}\right\} \Rightarrow (c_1\vec{u}_1 + \dots + c_p\vec{u}_p)\cdot\vec{v} = c_1\,\vec{u}_1\cdot\vec{v} + \dots + c_p\,\vec{u}_p\cdot\vec{v}$$

$\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ iff $\vec{u} = \vec{0}$

def Length (or "norm") of $\vec{v} \in \mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + \dots + v_n^2} \geq 0$

- $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

Ex: $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\|\vec{v}\| = \sqrt{a^2+b^2}$ = length of a line segment from $(0,0)$ to (a,b)



- $\|c\vec{v}\| = |c|\,\|\vec{v}\|$
- a vector of length 1 – "unit vector"

for $\vec{v} \neq \vec{0}$, $\vec{v} \longrightarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ - unit vector in the direction of $\vec{v}$

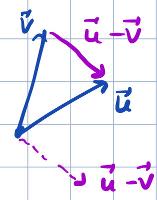
"normalizing" $\vec{v}$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ Q: find a unit vector $\vec{u}$ in the direction of $\vec{v}$

Sol: $\|\vec{v}\| = \sqrt{14} \Rightarrow \vec{u} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

def for $\vec{u}, \vec{v} \in \mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$ is

$$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$



Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\rightsquigarrow \vec{u} - \vec{v} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $dist(\vec{u}, \vec{v}) = \left\| \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right\| = \sqrt{13}$

Orthogonal vectors

def vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$
(perpendicular)

Notation: $\vec{u} \perp \vec{v}$

Rem: $\vec{0} \perp \vec{v}$ for any $\vec{v}$.

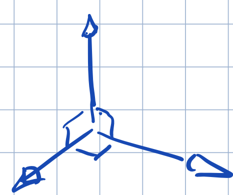
def A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal set if $\vec{u}_i \cdot \vec{u}_j = 0$ for each pair $i \neq j$.

Ex: $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ Q: check that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ - orthogonal set

Sol: $\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$ ✓

$\vec{u}_1 \cdot \vec{u}_3 = 3 \cdot \left(-\frac{1}{2}\right) + 1 \cdot (-2) + 1 \cdot \frac{7}{2} = 0$ ✓

$\vec{u}_2 \cdot \vec{u}_3 = (-1)\left(-\frac{1}{2}\right) + 2 \cdot (-2) + 1 \cdot \frac{7}{2} = 0$ ✓



Thm If $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is a lin. indep. set. Hence, S is a basis for span(S).

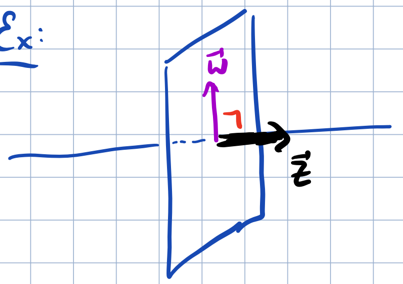
- An orthog. basis for a subspace $W \subset \mathbb{R}^n$ is a basis which is an orthog. set.

Orthogonal complements

def If $W \subset \mathbb{R}^n$ is a subspace and $\vec{z} \in \mathbb{R}^n$ is orthog. to each vector in W, then $\vec{z}$ is said to be orthogonal to W.

The set of all vectors in $\mathbb{R}^n$ orthogonal to W - "orthogonal complement of W",
notation: $W^\perp$.

Ex:



L - perpendicular line in $\mathbb{R}^3$

W - plane in $\mathbb{R}^3$

$L = W^\perp$ and $L^\perp = W$

For any $\underbrace{W \subset \mathbb{R}^n}_{\text{subspace}}$, $W^\perp \subset \mathbb{R}^n$ is a subspace. $\boxed{\dim W + \dim W^\perp = n}$

For A $m \times n$ matrix,
$$\boxed{\begin{array}{ll} (\text{null } A)^\perp = \text{row } A & \subset \mathbb{R}^n \\ (\text{col } A)^\perp = \text{null } A^T & \subset \mathbb{R}^m \end{array}}$$

Ex: Let $W = \text{span}\left(\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec w_1}, \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec w_2}\right) \subset \mathbb{R}^3$. Q: find a basis for $W^\perp$.

Sol: $W = \text{col}\left(\underbrace{\begin{bmatrix}1 & 1\\0 & 1\\0 & 1\end{bmatrix}}_{A}\right) \Rightarrow W^\perp = \text{null } A^T = \text{null}\left(\underbrace{\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\end{bmatrix}}_{B}\right)$

aug. mat. of $B\vec x = \vec 0$: $\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\1 & 1 & 1 & 0\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 1 & 0\end{array}\right]$ B

$x_1 \quad x_2 \quad \underset{\substack{\| \\ s}}{x_3}$ free

$x_1 = 0$
$x_2 = -s$
$x_3 = s$

$\vec x = s \underbrace{\begin{bmatrix}0\\-1\\1\end{bmatrix}}_{\vec v} \Rightarrow \{\vec v\}$ - basis for $W^\perp$.

Orthogonal projection onto a line

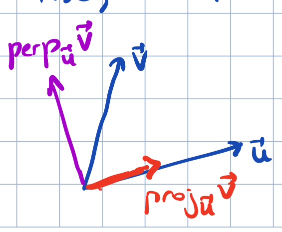
For $\vec v, \vec u \in \mathbb{R}^n$, $\vec u \neq \vec 0$, we can decompose $\vec v = \underbrace{\hat{\vec v}}_{\substack{\text{parallel} \\ \text{to } \vec u}} + \underbrace{\vec z}_{\substack{\text{orthog.} \\ \text{to } \vec u}}$

$\hat{\vec v} = \underset{\text{some coeff}}{\alpha} \vec u \quad \rightarrow \quad \vec v = \alpha \vec u + \vec z$

$\rightarrow \vec v \cdot \vec u = \alpha \vec u \cdot \vec u + 0 \quad \rightarrow \quad \alpha = \dfrac{\vec v \cdot \vec u}{\vec u \cdot \vec u} \quad \rightarrow \quad \hat{\vec v} = \dfrac{\vec v \cdot \vec u}{\vec u \cdot \vec u} \vec u =: \text{proj}_{\vec u} \vec v$

projection of $\vec v$ onto $\vec u$.

Also, denote $\vec z = \vec v - \text{proj}_{\vec u} \vec v =: \text{perp}_{\vec u} \vec v$



Thus: $\vec v = \underbrace{\text{proj}_{\vec u} \vec v}_{\parallel \vec u} + \underbrace{\text{perp}_{\vec u} \vec v}_{\perp \vec u}$

Rem $\text{proj}_{\vec{u}}\vec{v} = \text{proj}_{c\vec{u}}\vec{v}$ for any $c \neq 0$. So, it is actually a projection onto the line $L = \text{span}(\vec{u})$.

Ex: $\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Q: find $\text{proj}_{\vec{u}}\vec{v}$, $\text{perp}_{\vec{u}}\vec{v}$

Sol: $\text{proj}_{\vec{u}}\vec{v} = \dfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \dfrac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$; $\text{perp}_{\vec{u}}\vec{v} = \vec{v} - \text{proj}_{\vec{u}}\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Q: find the distance from $\vec{v}$ to $L = \text{span}(\vec{u})$.

Sol: $\text{dist}(\vec{v}, L) = \text{dist}(\vec{v}, \underbrace{\text{proj}_L\vec{v}}_{\text{closest point to } \vec{v} \text{ on } L}) = \|\text{perp}_L\vec{v}\| = \left\|\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\| = \sqrt{5}$