LAST TIME

(Zill 4.1.1)

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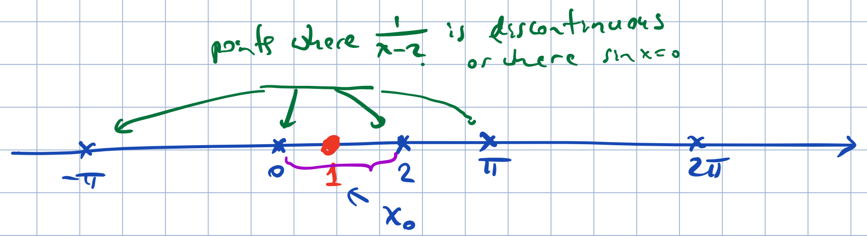
IVP for n^{th} order linear ODE:

$$\begin{cases} a_n(x)\dfrac{d^ny}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ y(x_0)=y_0 \,,\ y'(x_0)=y_1, \dots, y^{(n-1)}(x_0)=y_{n-1} \end{cases}$$

Thm (Existence and uniqueness for linear ODEs)

Let $a_n(x), \dots, a_0(x)$, $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every $x \in I$. If $x_0 \in I$, then a solution $y(x)$ of the IVP exists on I and is unique.

Ex:
$$\begin{cases} \sin(x)\,y'' + \dfrac{1}{x-2}\,y = 1 \\ y(1)=0 \,,\quad y'(1)=1 \end{cases}$$



So, sol. exists and is unique for $x \in (0,2)$.

linear ODEs	non-linear ODEs
possible points of discontinuity of the solution can be identified by finding points of discontinuity of coefficients $a_1 \dots a_n, g$ (and zeros of a_0)	← cannot be! (and can depend on the init. condition)

Ex: $\begin{cases} y' = y^2 \\ y(0)=1 \end{cases}$ $\rightsquigarrow \dfrac{dy}{y^2} = dx \Rightarrow -y^{-1} = x + C \Rightarrow y = \dfrac{-1}{x+C}$

$\rightsquigarrow 1 = \dfrac{-1}{0+C} \Rightarrow C = -1 \Rightarrow y = \dfrac{1}{1-x}$ solution

interval of existence $I = (-\infty, 1)$

point $x=1$ is not remarkable in any way from the eq.!

- can find the interval of existence without solving the eq. explicitly.

in fact: $\begin{cases} y' = y^2 \\ y(0)=y_0 \end{cases} \Rightarrow y = \dfrac{y_0}{1-y_0 x}$

interval of existence: $I = (-\infty, \frac{1}{y_0})$ if $y_0 > 0$,

$I = (\frac{1}{y_0}, \infty)$ if $y_0 < 0$.

Second order linear homogeneous ODEs (Zill 4.1.2)

n-th order ODE of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = \underline{\underline{0}} \qquad (*)$$

is said to be "homogeneous".

Thm (superposition principle)

Let $y_1, \dots, y_k$ be solutions of a homog. ODE on an interval I.
Then the linear combination $y = c_1 y_1(x) + \dots + c_k y_k(x)$, where $c_1, \dots, c_k$ are arbitrary constants, is also a solution on I.

Corollary a) if $y_1(x)$ is a sol. of $(*)$, a constant multiple $c y_1(x)$ is also a solution.

Ex: $y'' - y = 0$ has solutions $y_1 = e^x$, $y_2 = e^{-x}$

Thus $y = c_1 e^x + c_2 e^{-x}$ is a sol. for any c_1, c_2

- we are interested in a linearly independent set of solutions $\{y_1, \dots, y_n\}$ of $(*)$

def Let $f_1(x), \dots, f_n(x)$ be functions possessing at least $n-1$ derivatives each.
The determinant

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions.

Thm (criterion for linear independence of solutions)

Let $y_1, \dots, y_n$ be n solutions of n-th order linear ODE on an interval I.
Then the set of sol. is lin. independent on I iff $W(y_1, \dots, y_n) \neq 0$ for every $x \in I$.

In fact $W(y_1, \dots, y_n)$ is either zero everywhere on I or nonzero everywhere on I.

Abel's theorem: if $y_1, \dots, y_n$ - solutions of $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = 0$, then $W(y_1, \dots, y_n) = C e^{-\int \frac{a_{n-1}(x)}{a_n(x)}dx}$ for some constant C.

Ex: $y_1 = \sin x$, $y_2 = \cos x$ - two solutions of $y'' + y = 0$

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ (\sin x)' & (\cos x)' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0 \text{ for any } x.$$

So, y_1 and y_2 are lin. independent.

def Any set $y_1, \dots, y_n$ of n lin. indep. solutions of n^{th} order lin. homog. ODE $(*)$ on an interval I is said to be a fundamental set of solutions (FSS) on I.

Thm A fund. set of sol. of $(*)$ exists on an interval I (where a_i's are cont., $a_n \neq 0$)

Thm Let $y_1, \dots, y_n$ be a FSS of $(*)$ on I.
Then the general solution of $(*)$ on I is

$y = C_1 y_1(x) + \dots + C_n y_n(x)$ where $C_1, \dots, C_n$ are arbitrary constants.

Ex: $y_1 = x^{1/2}$, $y_2 = x^{-1}$ two sol. of $\boxed{2x^2 y'' + 3xy' - y = 0}$ on $I = (0, \infty)$

$$W(y_1, y_2) = \begin{vmatrix} x^{1/2} & x^{-1} \\ \frac{1}{2}x^{-1/2} & -x^{-2} \end{vmatrix} = -x^{-3/2} - \frac{1}{2}x^{-3/2} = -\frac{3}{2}x^{-3/2} \neq 0 \text{ for } x > 0$$

$\Rightarrow y_1, y_2$ - a FSS.

Thus, $y = C_1 x^{1/2} + C_2 x^{-1}$ - general sol. of on $(0, +\infty)$

Aside: consider the lin. transformation
(connection to lin. alg.)

functions on I with n cont. derivatives — cont. functions on I

$$L: C^n(I) \longrightarrow C^0(I) \quad \text{– "differential operator"}$$

with $L(y(x)) = a_n(x)\dfrac{d^n y}{dx^n} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)y$

Then $\{\text{solutions of } (*)\} = \ker(L)$ – subspace of $C^n(I)$ of dimension n

FSS = basis for $\ker(L)$.

Reduction of order (Zill 4.2)

2^{nd} order lin. homog. ODE $\quad a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (\#)$

Suppose we know one solution y_1. We want to look for a second, lin. indep from y_1, solution y_2 as $y_2 = u(x)y_1$. Substituting $y_2 = uy_1$ in $(\#)$, we find u.

Ex: $y'' - y = 0$, $y_1 = e^x$ is a sol. on $(-\infty, \infty)$. Use reduction of order to find a second sol. y_2.

Sol: $y = u(x)y_1 = u(x)e^x \to y' = u'e^x + ue^x$

$\to y'' = u''e^x + u'e^x + u'e^x + ue^x$
$= u''e^x + 2u'e^x + ue^x$

$\Rightarrow y'' - y = e^x(u'' + 2u') = 0$

$\xrightarrow{\cdot e^{-x}} u'' + 2u' = 0 \xrightarrow[u'=w]{\text{denote}} w' + 2w = 0 \to \dfrac{d}{dx}(e^{2x}w) = 0 \to$

linear 1^{st} order ODE, $\mu(x) = e^{2x}$ – int. factor

$\to w = C_1e^{-2x}$ or $u' = C_1e^{-2x}$

$\xrightarrow[\text{int. in } x]{} u = -\frac{1}{2}C_1e^{-2x} + C_2 \to y = uy_1 = -\frac{1}{2}C_1e^{-x} + C_2e^x \qquad (**)$

choose $C_1 = -2$, $C_2 = 0$: $y_2 = e^{-x}$

$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0 \Rightarrow \{y_1, y_2\}$ - FSS $\Rightarrow (**)$ is the general sol. of $y'' - y$ on $I = (-\infty, \infty)$