LAST TIME

Variation of parameters $y'' + P(x)y' + Q(x)y = f(x)$ (*)

y_1, y_2 - FSS of $y'' + P(x)y' + Q(x)y = 0$ (**)

Then $\boxed{Y = y_1(x)\int \frac{-y_2 f}{W}\,dx + y_2(x)\int \frac{y_1 f}{W}\,dx}$, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Green's function: $\boxed{G(x,t) = \frac{-y_1(x)y_2(t) + y_2(x)y_1(t)}{W(t)}}$

Then: $y(x) = \int_{x_0}^{x} G(x,t)f(t)\,dt$ - sol. of (*) with $y(x_0) = 0$, $y'(x_0) = 0$

exam 3 curve:

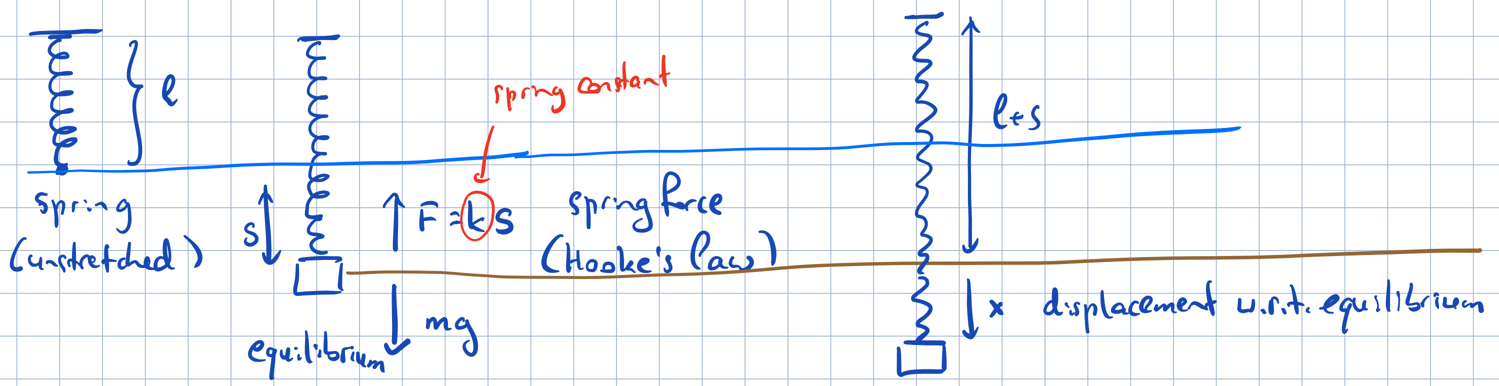
A	A−	B+	B	B−	C+	C	C−	D	F
93	90	87	83	80	77	73	69	50	0

median 85

avg 79.53

Vibrations (Zill 5.1)

Spring-mass system



$mg - ks = 0$

Newton's 2nd law: $m\dfrac{d^2x}{dt^2} = \underbrace{\overbrace{-k(x+s)}^{\text{spring force}} + mg}_{\text{net force}} = -kx$

$\xrightarrow{\cdot\frac{1}{m}}$ $\dfrac{d^2x}{dt^2} = -\underbrace{\boxed{\dfrac{k}{m}}}_{\omega^2} x$ – "undamped free motion" (or "simple harmonic motion")

General eq.:

$m x'' + \beta x' + kx = f(t)$

β: damping, $f(t)$: external force

Solutions: $x = c_1 \cos \omega t + c_2 \sin \omega t$

period of motion: $T = \dfrac{2\pi}{\omega}$

Ex: a mass weighing 2 lb stretches the spring by $s = 6$ in. At $t=0$, the mass is released from a point $x_0 = 8$ in $= \frac{2}{3}$ ft below equilibrium with initial upward velocity $v_0 = -\frac{4}{3}$ ft/s. Determine the eq. of motion.

Sol: in → ft: $s = \frac{1}{2}$ ft, $m = \dfrac{2\,lb}{g} = \dfrac{2\,lb}{32\,ft/s^2} = \dfrac{1}{16}\dfrac{lb\cdot s^2}{ft}$ = "slug"

equilibrium eq.: $\underbrace{k s}_{\frac{1}{2}ft} = \underbrace{mg}_{2\,lb} \Rightarrow k = 4\,lb/ft$ spring constant

$$\Rightarrow \begin{cases} \dfrac{d^2x}{dt^2} = \underbrace{-64}_{\omega^2} x \Rightarrow \omega = 8 \\ x(0) = \frac{2}{3},\ x'(0) = -\frac{4}{3} \end{cases}$$

$\Rightarrow$ $x = c_1 \cos 8t + c_2 \sin 8t$ $\quad (x' = -8c_1 \sin 8t + 8c_2 \cos 8t)$

$x(0) = c_1 = \frac{2}{3}$

$x'(0) = 8c_2 = -\frac{4}{3} \Rightarrow c_2 = -\frac{1}{6}$

$\Rightarrow x = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$

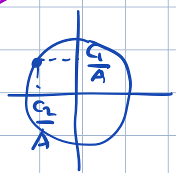
Rem $\quad c_1 \cos\omega t + c_2 \sin\omega t = \underbrace{A}_{\text{amplitude}} \sin(\omega t + \underbrace{\varphi}_{\text{phase angle}})$

where $A = \sqrt{c_1^2 + c_2^2}$, φ - determined from $\begin{cases} \sin\varphi = \dfrac{c_1}{A} \\ \cos\varphi = \dfrac{c_2}{A} \end{cases}$ $\quad \left(\Rightarrow \tan\varphi = \dfrac{c_1}{c_2}\right)$

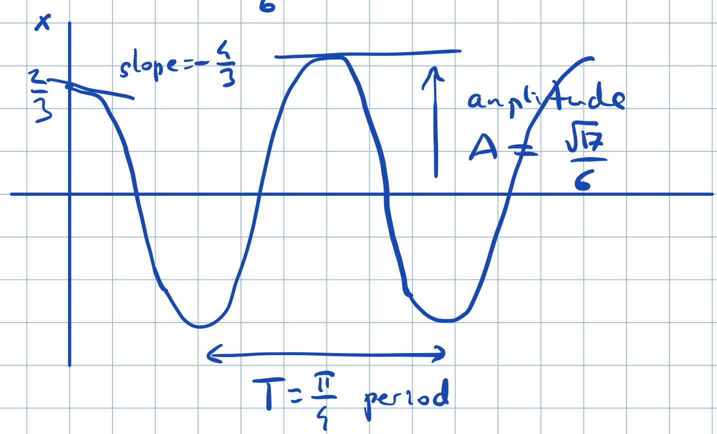
In Ex. above, $A = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \dfrac{\sqrt{17}}{6} \approx 0.69$ ft

$\varphi = \underbrace{\arctan(-4)}_{-1.326}$ or $\underbrace{\arctan(-4) + \pi}_{1.816} \simeq 1.816$ rad

in II quadrant since $c_1 > 0$, $c_2 < 0$



so: $x(t) = \dfrac{\sqrt{17}}{6} \sin(8t + 1.816\ldots)$ $\qquad$ period $T = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$ seconds



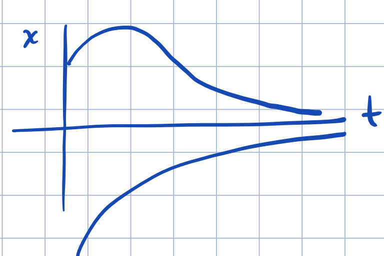
Free damped motion $\qquad m\dfrac{d^2x}{dt^2} = -kx - \underbrace{\overbrace{\beta \dfrac{dx}{dt}}^{\text{damping force}}}_{\beta:\ \text{damping constant}}$

$\rightsquigarrow \dfrac{d^2x}{dt^2} + \underbrace{\left(\dfrac{\beta}{m}\right)}_{2\lambda} \dfrac{dx}{dt} + \underbrace{\left(\dfrac{k}{m}\right)}_{\omega^2} x = 0$ $\quad$ - DE of free damped motion

aux. eq. $m^2 + 2\lambda m + \omega^2 = 0 \rightarrow m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

Case I: $\lambda^2 - \omega^2 > 0$ $\qquad x(t) = C_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + C_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$

(Overdamped motion - non-oscillating)



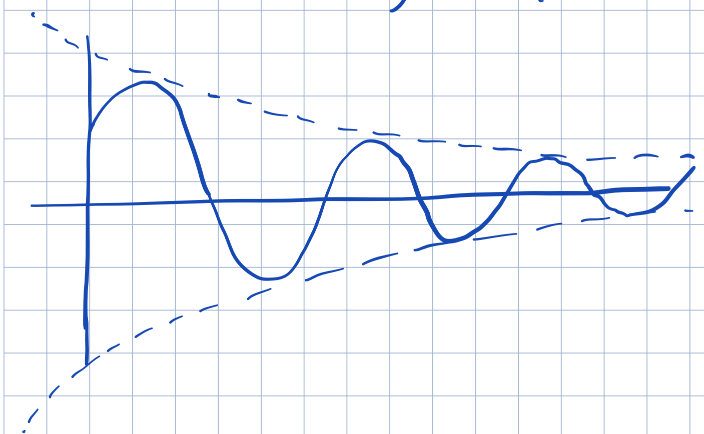
Case II (critically damped) $\quad \lambda^2-\omega^2=0 \Rightarrow x(t)=c_1 e^{-\lambda t}+c_2 t e^{-\lambda t}$

Case III $\quad \lambda^2-\omega^2<0 \Rightarrow x(t)=e^{-\lambda t}\left(c_1\cos\sqrt{\omega^2-\lambda^2}\,t+c_2\sin\sqrt{\omega^2-\lambda^2}\,t\right)$

(underdamped - oscillatory motion, with amplitude exponentially decreasing with time)

$\mu=\sqrt{\omega^2-\lambda^2}$ - "quasi-frequency"

$T_d=\frac{2\pi}{\mu}$ - "quasi-period"



Driven (forced) motion

$$m x''+\beta x'+kx=\underbrace{f(t)}_{\text{external force acting on the spring}}$$

$\cdot\frac{1}{m}$

$$x''+2\lambda x'+\omega^2 x=F(t)$$

Ex: $\begin{cases} x''+\omega^2 x=\overbrace{F_0\sin\gamma t}^{\text{ext. force}} \\ x(0)=0,\quad x'(0)=0\end{cases}$ (no damping), $\gamma\neq\omega$

F_0, γ ↑ some constants

complementary function: $x_c=c_1\cos\omega t+c_2\sin\omega t$

part. sol.: $X=A\cos\gamma t+B\sin\gamma t$ (undetermined coeff.)

$$X''+\omega^2X=(-A\gamma^2+\omega^2A)\cos\gamma t+(-B\gamma^2+\omega^2B)\sin\gamma t \overset{\text{WANT}}{=} F_0\sin\gamma t.$$

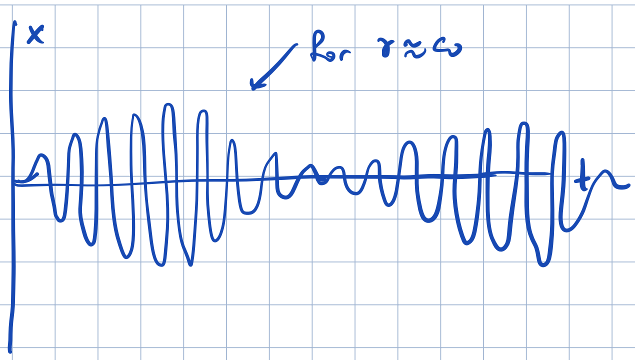
$\Rightarrow \begin{array}{l} A=0 \\ B=\frac{F_0}{\omega^2-\gamma^2}\end{array} \quad \Rightarrow X=\frac{F_0}{\omega^2-\gamma^2}\sin\gamma t$

$\Rightarrow x(t)=\frac{F_0}{\omega^2-\gamma^2}\sin\gamma t+c_1\cos\omega t+c_2\sin\omega t$

init. cond.: $x(0)=c_1=0$
$x'(0)=\frac{F_0\gamma}{\omega^2-\gamma^2}+c_2\omega=0$ $\Rightarrow c_1=0,\ c_2=-\frac{\gamma}{\omega}\frac{F_0}{\omega^2-\gamma^2}$

$\Rightarrow x(t)=\boxed{\frac{F_0}{\omega^2-\gamma^2}}\sin\gamma t-\boxed{\frac{\gamma}{\omega}\frac{F_0}{\omega^2-\gamma^2}}\sin\omega t$

amplitudes tend to $+\infty$ as $\gamma\to\omega$

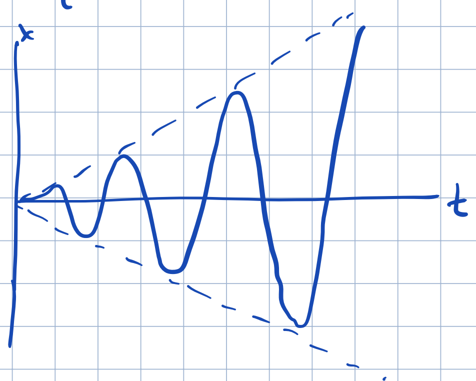


Resonance : case $\gamma = \omega$

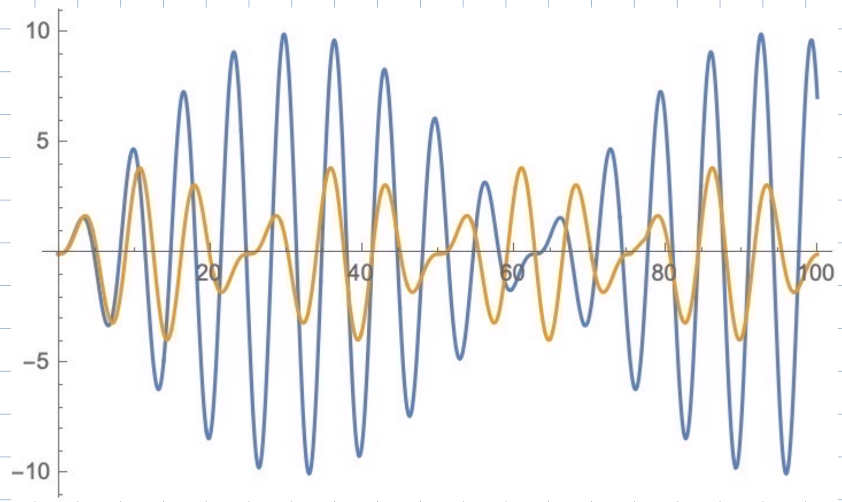
$$\begin{cases} x'' + \omega^2 x = F_0 \sin \omega t \\ x(0) = 0 \ , \ x'(0) \end{cases}$$

undetermined coeff.

$$x(t) = \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t$$



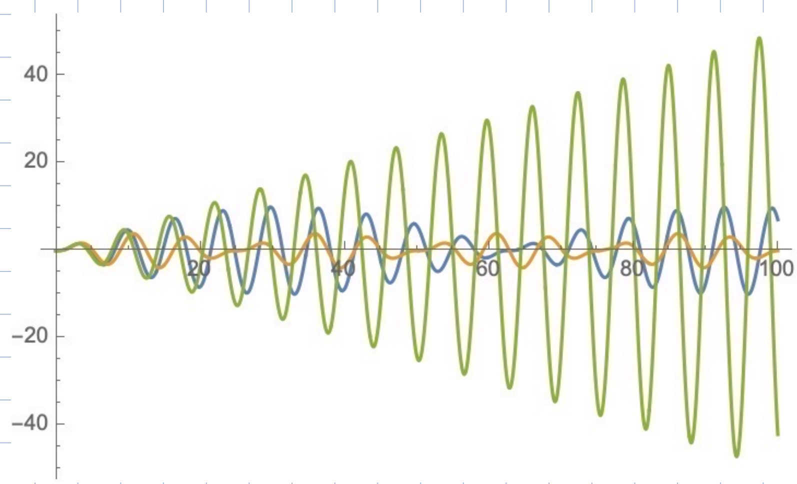
- oscillatory motion with growing amplitude.



$F_0 = 1$, $\omega = 1$

$\gamma = 0.75$, 0.9

(orange) (blue)



Green:

$\omega = \gamma = 1$