Reduction of order (Zill 4.2)

2nd order lin. homog. ODE $\quad a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (\#)$

Suppose we know one solution y_1. We want to look for a second, lin. indep from y_1, solution y_2 as $y_2 = u(x)\,y_1$. Substituting $y_2 = u\,y_1$ in $(\#)$, we find u.

Ex: $y'' - y = 0$, $y_1 = e^x$ is a sol. on $(-\infty, \infty)$. Use reduction of order to find a second sol. y_2.

Sol: $y = u(x)y_1 = u(x)e^x \rightarrow y' = u'e^x + ue^x$

$\rightarrow y'' = u''e^x + u'e^x + u'e^x + ue^x$
$= u''e^x + 2u'e^x + ue^x$

$\Rightarrow y'' - y = e^x(u'' + 2u') = 0$

$\xrightarrow{\cdot e^{-x}} u'' + 2u' = 0 \xrightarrow[\text{denote } u' = w]{} w' + 2w = 0 \rightarrow \frac{d}{dx}(e^{2x}w) = 0 \rightarrow$

linear 1st order ODE, $\mu(x) = e^{2x}$ - int. factor

$\rightarrow w = C_1 e^{-2x}$ or $u' = C_1 e^{-2x}$

$\xrightarrow[\text{int. in } x]{} u = -\frac{1}{2}C_1 e^{-2x} + C_2 \rightarrow y = u\,y_1 = -\frac{1}{2}C_1 e^{-x} + C_2 e^x \qquad (**)$

choose $C_1 = -2$, $C_2 = 0$: $y_2 = e^{-x}$

$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0 \Rightarrow \{y_1, y_2\}$ - FSS $\Rightarrow (**)$ is the general sol. of $y'' - y$ on $I = (-\infty, \infty)$

General case: $y'' + P(x)y' + Q(x)y = 0 \quad (*)$ $\qquad$ P, Q continuous on I.

Assume a sol. y_1 is known and $y_1(x) \neq 0$ for every $x \in I$.

Substitute $y = u(x)y_1$ into $(*)$:

$$y = uy_1$$
$$y' = uy_1' + u'y_1$$
$$y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$\Rightarrow y'' + Py' + Qy = u\overbrace{(y_1'' + Py_1' + Qy_1)}^{0} + u'(2y_1' + Py_1) + u''y_1$$

$$\Rightarrow y_1 u'' + (2y_1' + Py_1)u' = 0 \quad \underset{\text{denote } u' = w}{\Rightarrow}$$

$$\Rightarrow y_1 w' + (2y_1' + Py_1)w = 0 \quad \text{- linear and separable}$$

$$\sim \frac{dw}{w} = -\left(2\frac{y_1'}{y_1} + P\right)dx \sim \ln w = -2\ln y_1 - \int P(x)\,dx + C$$

$$\sim \underbrace{w}_{u'} = C_1 \frac{1}{y_1^2} e^{-\int P(x)\,dx} \Rightarrow u = C_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx + C_2$$

Setting $C_1 = 1$, $C_2 = 0$, we get

$$y_2 = uy_1 = \boxed{y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx}$$

Ex: $x^2y'' - 3xy' + 4y = 0$ has a sol. $y_1 = x^2$ on $I = (0, \infty)$

Find the general sol.

Sol: Standard form: $y'' \underbrace{- \frac{3}{x}}_{P(x)} y' + \frac{4}{x^2}y = 0$

$$y_2 = x^2 \int \frac{e^{-\int(-\frac{3}{x})dx}}{(x^2)^2}\,dx = x^2 \int \frac{\overbrace{e^{3\ln x}}^{= x^3}}{x^4}\,dx = x^2 \int \frac{dx}{x} = x^2 \ln x$$

So, general sol. on $(0, \infty)$ is: $y = C_1 x^2 + C_2 x^2 \ln x$

Linear homogeneous ODEs with constant coefficients (Zill 4.3)

2^{nd} order: $\underbrace{a\,y'' + b\,y' + c\,y}_{\text{constants}} = 0 \qquad (**)$

Try $y = e^{mx}$ (m: some constant) $\rightarrow y' = m e^{mx}$, $y'' = m^2 e^{mx}$

So, $(**) \rightarrow e^{mx}(am^2 + bm + c) = 0$. Thus, $y = e^{mx}$ is a sol. of $(**)$

iff $\boxed{am^2 + bm + c = 0}$ ← "auxiliary equation" (form) for ODE $(**)$.

Roots of auxiliary eq: $m_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Cases: (I) m_1 and m_2 are real and distinct (if $b^2 - 4ac > 0$)

(II) $m_1 = m_2 = -\dfrac{b}{2a}$ - repeated root (if $b^2 - 4ac = 0$)

(III) m_1 and m_2 are complex conjugate numbers (if $b^2 - 4ac < 0$)

(I) m_1, m_2 real, distinct $\Rightarrow y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$ - fund. set of solutions on $(-\infty, \infty)$

$\Rightarrow y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ - general sol. on $(-\infty, \infty)$

(II) Repeated real roots: $m_1 = m_2 = -\dfrac{b}{2a}$ $\Rightarrow y_1 = e^{m_1 x}$ - the only exponential solution.

We can find the second sol. from reduction of order:

$$y_2 = e^{m_1 x} \int \underbrace{\frac{e^{-\frac{b}{a}x}}{e^{2m_1 x}}}_{=1} dx = \boxed{x e^{m_1 x}}$$

$\Rightarrow$ general sol.: $y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

(III) Conjugate real roots

$m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ $\quad (\alpha, \beta > 0$ real$)$

$\leadsto$ as in case (I) $\quad y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha - i\beta)x}$. We can rewrite it more conveniently.

Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ $\Rightarrow y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) =$

$$= e^{\alpha x}\Big(\underbrace{(c_1+c_2)}_{\tilde{c}_1}\cos\beta x + \underbrace{i(c_1 - c_2)}_{\tilde{c}_2}\sin\beta x\Big)$$

So, the general solution of (**) is:

$$y = \tilde{c}_1 e^{\alpha x}\cos\beta x + \tilde{c}_2 e^{\alpha x}\sin\beta x$$

arbitrary constants

$y_1 = e^{\alpha x}\cos\beta x$, $\quad y_2 = e^{\alpha x}\sin\beta x$ $\quad$ – a fund. set of solutions of (**)