LAST TIME

Reduction of order $\quad \boxed{y'' + P(x)y' + Q(x)y = 0} \; (*)$, y_1 - solution

$$\Rightarrow \boxed{y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx} \quad \text{- also a solution.}$$

$\{y_1, y_2\}$ - FSS , $\quad y = C_1 y_1(x) + C_2 y_2(x)$ - general sol.

Homogeneous ODEs with constant coefficients

$$ay'' + by' + cy = 0$$

$y = e^{mx}$ - solution iff $\boxed{am^2 + bm + c = 0}$. roots: $m_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- auxiliary eq.

Case I $\quad m_1, m_2$ real, distinct

$\Rightarrow \{y_1 = e^{m_1 x}, y_2 = e^{m_2 x}\}$ - FSS , $\quad y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ - gen. sol.

Case II $\quad m_1 = m_2$ - repeated root. $\quad y_1 = e^{m_1 x}$

$\left(= -\dfrac{b}{2a}\right)$

from reduction of order

$$y_2 = e^{m_1 x} \int \underbrace{\frac{\overbrace{e^{-\int \frac{b}{a} dx}}^{= e^{-\frac{b}{a}x}}}{\underbrace{e^{2m_1 x}}_{= e^{-\frac{b}{a}x}}}}_{= 1} dx = x e^{m_1 x}$$

So: $\{y_1 = e^{m_1 x}, y_2 = x e^{m_1 x}\}$ - FSS , $\quad y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$ - gen. sol.

(III) Conjugate real roots

$m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ $(\alpha, \beta > 0$ real$)$

$\underset{\text{as in case (I)}}{\rightsquigarrow}$ $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha - i\beta)x}$. We can rewrite it more conveniently.

Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ $\Rightarrow$ $y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) =$

$$= e^{\alpha x}\Big(\underbrace{(c_1+c_2)}_{\tilde{c}_1}\cos\beta x + \underbrace{i(c_1-c_2)}_{\tilde{c}_2}\sin\beta x\Big)$$

So, the general solution of $(*)$ is:

$$y = \tilde{c}_1 e^{\alpha x}\cos\beta x + \tilde{c}_2 e^{\alpha x}\sin\beta x$$

arbitrary constants

$y_1 = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$ – a fund. set of solutions of $(*)$

Ex: a) $2y'' - 5y' - 3y = 0$

aux. eq.: $\underbrace{2m^2 - 5m - 3}_{(2m+1)(m-3)} = 0$ $\Rightarrow$ $m_1 = 3$, $m_2 = -\frac{1}{2}$

$$y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}$$

b) $y'' + 4y' + 4y = 0$

aux. eq.: $\underbrace{m^2 + 4m + 4}_{(m+2)^2} = 0$ $m_1 = m_2 = -2$ repeated root

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

c) $y'' + 4y' + 7y = 0$

aux. eq.: $m^2 + 4m + 7 = 0$ $m = \dfrac{-4 \pm \sqrt{4^2 - 4\cdot 7}}{2} = \dfrac{-4 \pm \sqrt{-12}}{2} = \underbrace{-2}_{\alpha} \pm i\underbrace{\sqrt{3}}_{\beta}$

$$\Rightarrow \quad y = c_1 e^{-2x}\cos(\sqrt{3}x) + c_2 e^{-2x}\sin(\sqrt{3}x)$$

Ex: Solve the IVP $\begin{cases} 2y'' - 5y' - 3y = 0 \\ y(0) = 0, \; y'(0) = 1 \end{cases}$

Sol: gen. sol. of the ODE: $y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}$

$\left(y' = 3c_1 e^{3x} - \frac{1}{2} c_2 e^{-\frac{1}{2}x} \right)$

$$\begin{matrix} y(0) = 0 \\ y'(0) = 1 \end{matrix} \Rightarrow \underbrace{\boxed{\begin{matrix} c_1 + c_2 = 0 \\ 3c_1 - \frac{1}{2} c_2 = 1 \end{matrix}}}_{\text{lin. sys. for } c_1, c_2} \rightsquigarrow \begin{matrix} c_1 = \frac{2}{7} \\ c_2 = -\frac{2}{7} \end{matrix} \Rightarrow y = \frac{2}{7} e^{3x} - \frac{2}{7} e^{-\frac{1}{2}x}$$

Ex: $y'' + k^2 y = 0$ (k real) $\Rightarrow m_{1,2} = \pm i \underbrace{k}_{\beta}$ (and $\alpha = 0$) $\Rightarrow y = c_1 \cos kx + c_2 \sin kx$

Ex: $y'' - k^2 y = 0 \Rightarrow m_{1,2} = \pm k \Rightarrow y = c_1 e^{kx} + c_2 e^{-kx}$

$$= \tilde{c}_1 \cosh kx + \tilde{c}_2 \sinh kx$$

$$\cosh kx = \frac{e^{kx} + e^{-kx}}{2} \qquad \sinh kx = \frac{e^{kx} - e^{-kx}}{2}$$

Nonhomogeneous second order linear equations

(*) $a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$ nonhomogeneous eq.

(**) $a_2(x) y'' + a_1(x) y' + a_0(x) y = \underline{\underline{0}}$ associated homogeneous eq.

Thm Let Y be any particular solution of the nonhomog. eq. (*) on an interval I, and let $\{y_1, y_2\}$ be a FSS of the associated homog. eq. (**) on I. Then the general sol. of nonhomog. eq. (*) is:

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{y_c(x) \text{ - "complementary function"}} + Y(x)$$

with c_1, c_2 arbitrary constants

Rem: General sol of NH = general sol of H + particular sol. of NH

- As in lin. alg.: for $T: V \to W$ linear transf.,

$$\{\vec{x} \in V \mid T(\vec{x}) = \underset{\substack{\uparrow \\ \text{given vector in } W}}{\vec{b}}\} = \{\vec{X} + \vec{x}_c \mid \vec{X} \text{ - some sol. of } T(\vec{x}) = \vec{b},\ \vec{x}_c \text{ - general sol. of } T(\vec{x}) = \vec{0}\}$$

How to find a particular solution Y?

- Method of undetermined coefficients.

  - assume a form of Y keeping some coefficients undetermined; obtain their values from substituting into (*).

- Assume that coeffs a_0, a_1, a_2 are constants:

$$ay'' + by' + cy = g(x)$$

Ex: $y'' - 3y' - 4y = 3e^{2x}$ (#) find Y

Sol: try looking for Y of the form $Y = \underset{\text{coeff. to be determined}}{A}e^{2x} \to Y' = 2Ae^{2x},\ Y'' = 4Ae^{2x}$

$$\Rightarrow Y'' - 3Y' - 4Y = (\underbrace{4A - 6A - 4A}_{-6A})e^{2x} \overset{\text{WANT}}{=} 3e^{2x} \Rightarrow A = -\frac{1}{2}$$

$\Rightarrow \boxed{Y = -\frac{1}{2}e^{2x}}$ a solution

- complementary function: $y_c'' - 3y_c' - 4y_c = 0 \rightsquigarrow m^2 - 3m - 4 = 0$ (aux eq.) $\to m_1 = 4,\ m_2 = -1$

$\rightsquigarrow y_c = c_1 e^{4x} + c_2 e^{-x}$ - complementary function

$\to$ gen. sol of (#): $y = y_c + Y = \boxed{c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}e^{2x}}$

Ex: $y'' - 3y' - 4y = 2\sin x$

Sol: try $Y = A\sin x \to Y' = A\cos x,\ Y'' = -A\sin x$

$$\to Y'' - 3Y' - 4Y = (\underbrace{-A - 4A}_{-5A})\sin x - 3A\cos x \overset{\text{WANT}}{=} 2\sin x$$

$\left.\begin{array}{l} x = 0 \to -3A = 0 \\ x = \frac{\pi}{2} \to -5A = 2 \end{array}\right\}$ inconsistent! $\Rightarrow$ cannot find A s.t. $Y = A\sin x$ is a sol.

Next try: $Y = A\sin x + B\cos x$

$\Rightarrow$ $Y' = A\cos x - B\sin x$

$Y'' = -A\sin x - B\cos x$

$$Y'' - 3Y' - 4Y = \underbrace{(-A + 3B - 4A)}_{-5A+3B}\sin x + \underbrace{(-B - 3A - 4B)}_{-3A-5B}\cos x \overset{\text{WANT}}{=} 2\sin x$$

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0 \end{cases} \Rightarrow \begin{cases} A = -5/17 \\ B = 3/17 \end{cases} \Rightarrow Y = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$$

- for $g(x)$ a polynomial, try Y a polynomial of the same degree.

Ex: $y'' - 3y' - 4y = 4x^2 - 1$ $\rightsquigarrow$ try $Y = Ax^2 + Bx + C$

$\Rightarrow$ $Y' = 2Ax + B$

$Y'' = 2A$

So, $Y'' - 3Y' - 4Y = -4Ax^2 + (-6A - 4B)x + (2A - 3B - 4C) \underset{\text{WANT}}{=} 4x^2 - 1$

$$\begin{cases} -4A = 4 \\ -6A - 4B = 0 \\ 2A - 3B - 4C = -1 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 3/2 \\ C = -11/8 \end{cases} \Rightarrow Y = -x^2 + \frac{3}{2}x - \frac{11}{8}$$

Thus:
- for $g(x) = e^{\alpha x}$, try $Y = Ae^{\alpha x}$
- for $g(x) = \sin\beta x$, try $Y = A\sin\beta x + B\cos\beta x$
- for $g(x)$ a polynomial of degree n, try Y a poly. of degree n.

- for g a product of two or three of these types of functions, take a product of these forms of Y

Ex: $y'' - 3y' - 4y = -8e^x \cos 2x$

Sol: Try $Y = Ae^x \cos 2x + Be^x \sin 2x$

$$Y' = (A+2B)e^x \cos 2x + (B-2A)e^x \sin 2x$$

$$Y'' = \underbrace{(A+2B+2B-4A)}_{-3A+4B} e^x \cos 2x + \underbrace{(B-2A-2A-4B)}_{-4A-3B} e^x \sin 2x$$

$$Y'' - 3Y' - 4Y = \Big(\underbrace{(-3A+4B) - 3(A+2B) - 4A}_{-10A-2B}\Big) e^x \cos 2x +$$

$$+ \Big(\underbrace{(-4A-3B) - 3(-2A+B) - 4B}_{2A-10B}\Big) e^x \sin 2x \overset{\text{WANT}}{=} -8e^x \cos 2x$$

$$\begin{cases} -10A - 2B = -8 \\ 2A - 10B = 0 \end{cases} \longrightarrow \begin{array}{l} A = \frac{10}{13} \\ B = \frac{2}{13} \end{array} \Rightarrow Y = \frac{10}{13} e^x \cos 2x + \frac{2}{13} e^x \sin 2x$$