Towards perturbative topological field theory on manifolds with boundary

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Plan of the talk

- Background: topological field theory
- Hidden algebraic structure on cohomology of simplicial complexes coming from TFT
- One-dimensional simplicial Chern-Simons theory
- Topological field theory on manifolds with boundary

Axioms of an n-dimensional topological quantum field theory. (Atiyah'88)

Data:

1. To a closed $(n-1)$-dimensional manifold B a TFT associates a vector space $\mathcal{H}_B$ (the "space of states").
2. To a n-dimensional cobordism $\Sigma : B_1 \to B_2$ a TFT associates a linear map $Z_\Sigma : \mathcal{H}_{B_1} \to \mathcal{H}_{B_2}$ (the "partition function").
3. Representation of $\mathrm{Diff}(B)$ on $\mathcal{H}_B$.

Axioms:

(a) Multiplicativity "$\sqcup \to \otimes$":

$$\mathcal{H}_{B_1 \sqcup B_2} = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}, \quad Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$$

(b) Gluing axiom: for cobordisms $\Sigma_1 : B_1 \to B_2$, $\Sigma_2 : B_2 \to B_3$,

$$Z_{\Sigma_1 \cup_{B_2} \Sigma_2} = Z_{\Sigma_2} \circ Z_{\Sigma_1}$$

(c) Normalization: $\mathcal{H}_\varnothing = \mathbb{C}$.

(d) Diffeomorphisms of Σ constant on $\partial\Sigma$ do not change Z_Σ. Under general diffeomorphisms, Z_Σ transforms equivariantly.

Remarks:

- A closed n-manifold Σ can be viewed as a cobordism $\varnothing \xrightarrow{\Sigma} \varnothing$, so $Z_\Sigma : \mathbb{C} \to \mathbb{C}$ is a multiplication by a complex number – a diffeomorphism invariant of Σ.
- An n-TFT $(\mathcal{H}, Z)$ is a functor of symmetric monoidal categories $\mathrm{Cob}_n \to \mathrm{Vect}_\mathbb{C}$, with diffeomorphisms acting by natural transformations.

Reference: M. Atiyah, *Topological quantum field theories,* Publications Mathématiques de l'IHÉS, 68 (1988) 175–186.

A. S. Schwarz'78: path integral of the form

$$Z_\Sigma = \int_{F_\Sigma} \mathcal{D}X \; e^{\frac{i}{\hbar}S(X)}$$

with S a local functional on F_Σ (a space of sections of a sheaf over Σ), invariant under $\mathrm{Diff}(\Sigma)$, can produce a topological invariant of Σ (when it can be defined, e.g. through formal stationary phase expression at $\hbar \to 0$).

Example: Let Σ be odd-dimensional, closed, oriented; let E be an acyclic local system, $F_\Sigma = \Omega^r(\Sigma, E) \oplus \Omega^{\dim \Sigma - r - 1}(\Sigma, E^*)$ with $0 \leq r \leq \dim \Sigma - 1$, and with the action

$$S = \int_\Sigma \langle b \overset{\wedge}{,} da \rangle$$

The corresponding path integral can be defined and yields the **Ray-Singer torsion** of Σ with coefficients in E.

Reference: A. S. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants,* Lett. Math. Phys. 2, 3 (1978) 247–252.

Witten'89: Let Σ be a compact, oriented, framed 3-manifold, G – a compact Lie group, $P = \Sigma \times G$ the trivial G-bundle over Σ. Set $F_\Sigma = \mathrm{Conn}(P) \simeq \mathfrak{g} \otimes \Omega^1(\Sigma)$ – the space of connections in P; $\mathfrak{g} = \mathrm{Lie}(G)$. For A a connection, set

$$S_{CS}(A) = \mathrm{tr}_\mathfrak{g} \int_\Sigma \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A$$

– the integral of the Chern-Simons 3-form. Consider

$$Z_\Sigma(k) = \int_{\mathrm{Conn}(P)} \mathcal{D}A \; e^{\frac{ik}{2\pi} S_{CS}(A)}$$

for $k = 1, 2, 3, \ldots$ (i.e. $\hbar = \frac{2\pi}{k}$). For closed manifolds, $Z(\Sigma, k)$ is an interesting invariant, calculable explicitly through surgery. E.g. for $G = SU(2)$, $\Sigma = S^3$, the result is

$$Z_{S^3}(k) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right)$$

The space of states $\mathcal{H}_B$ corresponding to a surface B is the geometric quantization of the moduli space of local systems $\mathrm{Hom}(\pi_1(B), G)/G$ with Atiyah-Bott symplectic structure.
For a **knot** $\gamma : S^1 \hookrightarrow \Sigma$, Witten considers the expectation value

$$W(\Sigma, \gamma, k) = Z_\Sigma(k)^{-1} \int_{\mathrm{Conn}(P)} \mathcal{D}A \, e^{\frac{ik}{2\pi} S_{CS}(A)} \, \mathrm{tr}_R \, \mathrm{hol}(\gamma^* A)$$

where R is a representation of G. In case $G = SU(2)$, $\Sigma = S^3$, this expectation value yields the value of Jones' polynomial of the knot at the point $q = e^{\frac{i\pi}{k+2}}$.
Reference: E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 (1989), 351–399.

Axelrod-Singer'94: Perturbation theory (formal stationary phase expansion at $\hbar \to 0$) for Chern-Simons theory on a **closed**, oriented, framed 3-manifold rigorously constructed.

$$Z^{\text{pert}}_\Sigma(A_0,\hbar)=e^{\frac{i}{\hbar}S_{CS}(A_0)}\;\tau(\Sigma,A_0)\;e^{\frac{i\pi}{2}\eta(\Sigma,A_0,g)}\;e^{ic(\hbar)S_{\text{grav}}(g)}.$$

$$\cdot\exp\left(\frac{i}{\hbar}\sum_{\text{connected 3-valent graphs }\Gamma}\frac{(i\hbar)^{l(\Gamma)}}{|\text{Aut}(\Gamma)|}\int_{\text{Conf}_{V(\Gamma)}(\Sigma)}\prod_{\text{edges}}\pi^*_{e_1e_2}\eta\right)$$

where

- A_0 is a fixed **acyclic** flat connection, g is an arbitrary Riemannian metric,
- $\tau(\Sigma,A_0)$ is the Ray-Singer torsion, $\eta(\Sigma,A_0,g)$ is the Atiyah's eta-invariant,
- $V(\Gamma)$ and $l(\Gamma)$ are the number of vertices and the number of loops of a graph,
- $\text{Conf}_n(\Sigma)$ is the Fulton-Macpherson-Axelrod-Singer compactification of the configuration space of n-tuple distinct points on Σ,
- $\eta\in\Omega^2(\text{Conf}_2(\Sigma))$ is the **propagator**, a parametrics for the Hodge-theoretic inverse of de Rham operator, $d/(dd^*+d^*d)$, $\pi_{ij}:\text{Conf}_n(\Sigma)\to\text{Conf}_2(\Sigma)$ – forgetting all points except i-th and j-th.
- $S_{\text{grav}}(g)$ is the Chern-Simons action evaluated on the Levi-Civita connection, $c(\hbar)\in\mathbb{C}[[\hbar]]$.

Remarks:

- Expression for $\log Z$ is **finite** in each order in $\hbar$: given as a finite sum of integrals of smooth forms over compact manifolds.
- Propagator depends on the choice of metric g, but the whole expression does not depend on g.

Reference: S. Axelrod, I. M. Singer, *Chern-Simons perturbation theory. I.* Perspectives in mathematical physics, 17–49, Conf. Proc. Lecture Notes Math. Phys., III, Int. Press, Cambridge, MA (1994); *Chern-Simons perturbation theory. II.* J. Differential Geom. 39, 1 (1994) 173–213.

Comments:

- Explicit examples of Atiyah's 3-TFTs were constructed by Reshetikhin-Turaev'91 and Turaev-Viro'92 from representation theory of quantum groups at roots of unity.
- Main motivation to study TFTs is that they produce invariants of manifolds and knots.
- Example of a different application: use of the 2-dimensional Poisson sigma model on a disc in Kontsevich's deformation quantization of Poisson manifolds (Kontsevich'97, Cattaneo-Felder'00).

Problems:

1. Witten's treatment of Chern-Simons theory is not completely mathematically transparent (use of path integral as a "black box" which is assumed to have certain properties); Axelrod-Singer's treatment is transparent, but restricted to closed manifolds: perturbative Chern-Simons theory as Atiyah's TFT is not yet constructed.
2. Reshetikhin-Turaev invariants are conjectured to coincide asymptotically with the Chern-Simons partition function.
3. Construct a combinatorial model of Chern-Simons theory on triangulated manifolds, retaining the properties of a perturbative gauge theory and yielding the same manifold invariants.

Program/logic of the exposition:

Simplicial BF theory (P.M.)

($\longrightarrow$ hidden algebraic structure on cohomology of simplicial complexes)

$\downarrow$

One-dimensional simplicial Chern-Simons theory

(with A. Alekseev)

$\downarrow$

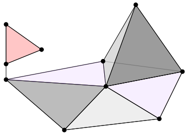
Perturbative TFT on manifolds with boundary

($\longrightarrow$ Euler-Lagrange moduli spaces: supergeometric structures, gluing, cohomological quantization. Gluing formulae for quantum invariants.)

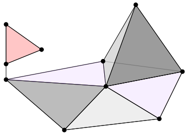
(partially complete, with A. Cattaneo and N. Reshetikhin)

$\downarrow$

Perturbative TFT on manifolds with corners (in progress)



Simplicial complex T



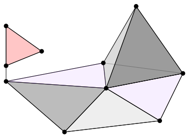
Simplicial complex T

$$\downarrow$$

Simplicial cochains $C^0(T) \to \cdots \to C^{\text{top}}(T)$,

$$C^k(T) = \text{Span}\{k-\text{simplices}\},$$

$$d_k : C^k(T) \to C^{k+1}(T), \qquad \underbrace{e_\sigma}_{\text{basis cochain}} \quad \mapsto \sum_{\sigma' \in T:\ \sigma \in \text{faces}(\sigma')} \pm e_{\sigma'}$$



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Cohomology $H^\bullet(T)$, $H^k(T) = \ker d_k \,/\, \mathrm{im}\, d_{k-1}$

— a homotopy invariant of T

Cohomology carries a commutative ring structure, coming from (non-commutative) Alexander's product for cochains.

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Massey operations on cohomology are a complete invariant of rational homotopy type in simply connected case (Quillen-Sullivan), i.e. rationalized homotopy groups $\mathbb{Q} \otimes \pi_k(T)$ can be recovered from them.

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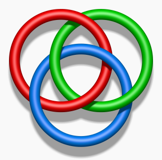

Example of use: linking of Borromean rings is detected by a non-vanishing Massey operation on cohomology of the complement.
$m_3([\alpha],[\beta],[\gamma]) = [u \wedge \gamma + \alpha \wedge v] \in H^2$
where $[\alpha],[\beta],[\gamma] \in H^1$, $du = \alpha \wedge \beta$, $dv = \beta \wedge \gamma$.

Another example: **nilmanifold**

$$M = \mathsf{H}_3(\mathbb{R})/\mathsf{H}_3(\mathbb{Z})$$
$$= \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R} \right\} \Big/ \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}$$

Denote

$$\alpha = dx,\ \beta = dy,\ u = dz - y\,dx \in \Omega^1(M)$$

Important point: $\alpha \wedge \beta = du$. The cohomology is spanned by classes

$$\underbrace{[1]}_{\text{degree } 0},\quad \underbrace{[\alpha],[\beta]}_{\text{degree } 1},\quad \underbrace{[\alpha\wedge u],[\beta\wedge u]}_{\text{degree } 2},\quad \underbrace{[\alpha\wedge\beta\wedge u]}_{\text{degree } 3}$$

and

$$m_3([\alpha],[\beta],[\beta]) = [u\wedge\beta] \in H^2(M)$$

is a non-trivial Massey operation.

Fix $\mathfrak{g}$ a unimodular Lie algebra (i.e. with $\mathrm{tr}[x, \bullet] = 0$ for any $x \in \mathfrak{g}$).

Main construction (P.M.)

Simplicial complex T

$\downarrow$ local formula

Unimodular L_∞ algebra structure on $\mathfrak{g} \otimes C^\bullet(T)$

$\downarrow$ homotopy transfer

Unimodular L_∞ algebra structure on $\mathfrak{g} \otimes H^\bullet(T)$

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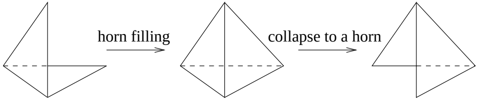
Main theorem (P.M.)

Unimodular L_∞ algebra structure on $\mathfrak{g} \otimes H^\bullet(T)$ (up to isomorphisms) is an invariant of T under simple homotopy equivalence.



Main construction (P.M.)

Simplicial complex T

$$\Big\downarrow \text{local formula}$$

Unimodular L_∞ algebra structure on $\mathfrak{g} \otimes C^\bullet(T)$

$$\Big\downarrow \text{homotopy transfer}$$

Unimodular L_∞ algebra structure on $\mathfrak{g} \otimes H^\bullet(T)$

- Thom's problem: lifting ring structure on $H^\bullet(T)$ to a **commutative** product on cochains. Removing $\mathfrak{g}$, we get a homotopy commutative algebra on $C^\bullet(T)$. This is an improvement of Sullivan's result with cDGA structure on cochains $= \Omega_{\text{poly}}(T)$.
- **Local** formulae for Massey operations.
- Our invariant is strictly stronger than rational homotopy type.

References:

- P. Mnev, *Discrete BF theory*, arXiv:0809.1160
- P. Mnev, *Notes on simplicial BF theory*, Moscow Mathematical Journal 9, 2 (2009), 371–410

Definition

A unimodular L_∞ algebra is the following collection of data:

(a) a $\mathbb{Z}$-graded vector space $V^\bullet$,

(b) "classical operations" $l_n : \wedge^n V \to V$, $n \geq 1$,

(c) "quantum operations" $q_n : \wedge^n V \to \mathbb{R}$, $n \geq 1$,

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subject to two sequences of quadratic relations:

1. $\sum_{r+s=n} \frac{1}{r!s!} l_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$, $n \geq 1$ (anti-symmetrization over inputs implied),
2. $\frac{1}{n!} \mathrm{Str}\, l_{n+1}(\bullet, \cdots, \bullet, -) + \sum_{r+s=n} \frac{1}{r!s!} q_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$

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Note:

- First classical operation satisfies $(l_1)^2 = 0$, so $(V^\bullet, l_1)$ is a complex.
- A unimodular L_∞ algebra is in particular an L_∞ algebra (as introduced by Lada-Stasheff), by ignoring q_n.
- Unimodular Lie algebra is the same as unimodular L_∞ algebra with $l_{\neq 2} = q_\bullet = 0$.

An alternative definition

A unimodular L_∞ algebra is a graded vector space V endowed with

- a vector field Q on $V[1]$ of degree 1,
- a function ρ on $V[1]$ of degree 0,

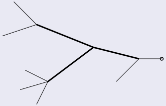
satisfying the following identities:

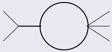
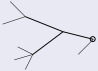
$$[Q, Q] = 0, \qquad \mathrm{div}\, Q = Q(\rho)$$

Homotopy transfer theorem (P.M.)

If $(V, \{l_n\}, \{q_n\})$ is a unimodular L_∞ algebra and $V' \hookrightarrow V$ is a deformation retract of (V, l_1), then

1. V' carries a unimodular L_∞ structure given by

$l'_n = \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to V'$

$q'_n = \sum_{\Gamma_1} \frac{1}{|\mathrm{Aut}(\Gamma_1)|}$  $+ \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to \mathbb{R}$

where Γ_0 runs over rooted trees with n leaves and Γ_1 runs over 1-loop graphs with n leaves.

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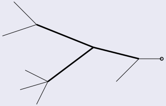
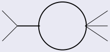
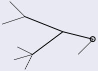
$l'_n = \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to V'$

$q'_n = \sum_{\Gamma_1} \frac{1}{|\mathrm{Aut}(\Gamma_1)|}$  $+ \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to \mathbb{R}$

where Γ_0 runs over rooted trees with n leaves and Γ_1 runs over 1-loop graphs with n leaves. **Decorations:**

leaf	$i : V' \hookrightarrow V$	root	$p : V \twoheadrightarrow V'$
edge	$-s : V^\bullet \to V^{\bullet-1}$	$(m+1)$-valent vertex	l_m
cycle	super-trace over V	m-valent $\circ$-vertex	q_m

where s is a chain homotopy, $l_1\, s + s\, l_1 = \mathrm{id} - i\, p$.

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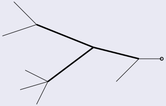
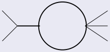
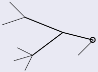
$l'_n = \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to V'$

$q'_n = \sum_{\Gamma_1} \frac{1}{|\mathrm{Aut}(\Gamma_1)|}$  $+ \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to \mathbb{R}$

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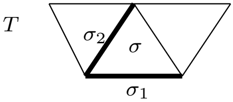
2. Algebra $(V', \{l'_n\}, \{q'_n\})$ changes by isomorphisms under changes of induction data (i, p, s).

Locality of the algebraic structure on simplicial cochains

$$l_n^T(X_{\sigma_1}e_{\sigma_1},\cdots,X_{\sigma_n}e_{\sigma_n}) = \sum_{\sigma\in T\,:\,\sigma_1,\ldots,\sigma_n\in\mathrm{faces}(\sigma)} \bar{l}_n^\sigma(X_{\sigma_1}e_{\sigma_1},\cdots,X_{\sigma_n}e_{\sigma_n})e_\sigma$$

$$q_n^T(X_{\sigma_1}e_{\sigma_1},\cdots,X_{\sigma_n}e_{\sigma_n}) = \sum_{\sigma\in T\,:\,\sigma_1,\ldots,\sigma_n\in\mathrm{faces}(\sigma)} \bar{q}_n^\sigma(X_{\sigma_1}e_{\sigma_1},\cdots,X_{\sigma_n}e_{\sigma_n})$$

Notations: e_σ – basis cochain for a simplex σ, $X_\bullet\in\mathfrak{g}$, $Xe_\sigma := X\otimes e_\sigma$.



Here $\bar{l}_n^\sigma : \wedge^n(\mathfrak{g}\otimes C^\bullet(\sigma))\to\mathfrak{g}$, $\bar{q}_n^\sigma : \wedge^n(\mathfrak{g}\otimes C^\bullet(\sigma))\to\mathbb{R}$ are universal local building blocks, depending on dimension of σ only, not on combinatorics of T.

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$\bar{l}_2(Xe_A, Ye_A) = [X, Y]$, all other operations vanish.

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One-dimensional simplex $\sigma = [AB]$:

$$\bar{l}_{n+1}(X_1 e_{AB}, \cdots, X_n e_{AB}, Y e_B) = \frac{B_n}{n!} \sum_{\theta \in S_n} [X_{\theta_1}, \cdots, [X_{\theta_n}, Y] \cdots]$$

$$\bar{l}_{n+1}(X_1 e_{AB}, \cdots, X_n e_{AB}, Y e_A) = (-1)^{n+1} \frac{B_n}{n!} \sum_{\theta \in S_n} [X_{\theta_1}, \cdots, [X_{\theta_n}, Y] \cdots]$$

$$\bar{q}_n(X_1 e_{AB}, \cdots, X_n e_{AB}) = \frac{B_n}{n \cdot n!} \sum_{\theta \in S_n} \mathrm{tr}_{\mathfrak{g}} \, [X_{\theta_1}, \cdots, [X_{\theta_n}, \bullet] \cdots]$$

where $B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,\ B_4 = -1/30, \ldots$ are Bernoulli numbers.

Higher-dimensional simplices, $\sigma = \Delta^N$**,** $N \geq 2$**:** $\bar{l}_n, \bar{q}_n$ are given by a *regularized* homotopy transfer formula for transfer $\mathfrak{g} \otimes \Omega^\bullet(\Delta^N) \to \mathfrak{g} \otimes C^\bullet(\Delta^N)$

Higher-dimensional simplices, $\sigma = \Delta^N$, $N \geq 2$: $\bar{l}_n, \bar{q}_n$ are given by a *regularized* homotopy transfer formula for transfer $\mathfrak{g} \otimes \Omega^\bullet(\Delta^N) \to \mathfrak{g} \otimes C^\bullet(\Delta^N)$, with

- $i=$ representation of cochains by Whitney elementary forms,
- $p=$ integration over faces,
- $s =$ Dupont's chain homotopy operator.

Higher-dimensional simplices, $\sigma=\Delta^N$, $N\geq 2$: $\bar{l}_n, \bar{q}_n$ are given by a *regularized* homotopy transfer formula for transfer $\mathfrak{g}\otimes\Omega^\bullet(\Delta^N)\to\mathfrak{g}\otimes C^\bullet(\Delta^N)$, with

- $i=$ representation of cochains by Whitney elementary forms,
- $p=$ integration over faces,
- $s=$ Dupont's chain homotopy operator.

$$\left.\begin{array}{c}\bar{l}_n^\sigma\\ \bar{q}_n^\sigma\end{array}\right\}(X_{\sigma_1}e_{\sigma_1},\cdots,X_{\sigma_n}e_{\sigma_n})=\sum_\Gamma C(\Gamma)^\sigma_{\sigma_1\cdots\sigma_n}\mathrm{Jacobi}_\mathfrak{g}(\Gamma;X_{\sigma_1},\cdots,X_{\sigma_n})$$

where Γ runs over **binary** rooted trees with n leaves for $\bar{l}_n^\sigma$ and over **trivalent** 1-loop graphs with n leaves for $\bar{q}_n^\sigma$;
$C(\Gamma)^\sigma_{\sigma_1\cdots\sigma_n}\in\mathbb{R}$ are structure constants.

Higher-dimensional simplices, $\sigma = \Delta^N$**,** $N \geq 2$**:** $\bar{l}_n, \bar{q}_n$ are given by a *regularized* homotopy transfer formula for transfer $\mathfrak{g} \otimes \Omega^\bullet(\Delta^N) \to \mathfrak{g} \otimes C^\bullet(\Delta^N)$, with

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$$\left.\begin{array}{c}\bar{l}_n^\sigma \\ \bar{q}_n^\sigma\end{array}\right\}(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n}) = \sum_\Gamma C(\Gamma)^\sigma_{\sigma_1\cdots\sigma_n} \mathrm{Jacobi}_\mathfrak{g}(\Gamma; X_{\sigma_1}, \cdots, X_{\sigma_n})$$

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$C(\Gamma)^\sigma_{\sigma_1\cdots\sigma_n} \in \mathbb{R}$ are structure constants.
There are explicit formulae for structure constants for small n.

Summary: logic of the construction

building blocks $\bar{l}_n, \bar{q}_n$ on Δ^N

$$\big\downarrow \text{combinatorics of } T$$

algebraic structure on cochains

$$\big\downarrow \text{homotopy transfer}$$

algebraic structure on cohomology

Summary: logic of the construction

$$\begin{array}{c} \text{building blocks } \bar{l}_n, \bar{q}_n \text{ on } \Delta^N \\ \Big\downarrow \text{combinatorics of } T \\ \text{algebraic structure on cochains} \\ \Big\downarrow \text{homotopy transfer} \\ \text{algebraic structure on cohomology} \end{array}$$

- Operations l_n on $\mathfrak{g} \otimes H^\bullet(T)$ are **Massey brackets** on cohomology and are a complete invariant of **rational homotopy type** in simply-connected case.
- Operations q_n on $\mathfrak{g} \otimes H^\bullet(T)$ give a version of **Reidemeister torsion** of T.
- Construction above yields new local combinatorial formulae for Massey brackets (in other words: Massey brackets lift to a local algebraic structure on simplicial cochains).

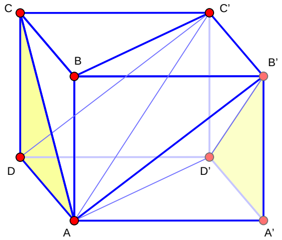
Example: for a circle and a Klein bottle, $H^\bullet(S^1) \simeq H^\bullet(KB)$ as rings, but $\mathfrak{g}\otimes H^\bullet(S^1) \not\simeq \mathfrak{g}\otimes H^\bullet(KB)$ as unimodular L_∞ algebras (distinguished by quantum operations).

$$e^{\sum_n \frac{1}{n!} q_n(X\otimes\varepsilon, \cdots X\otimes\varepsilon)} =$$

$\det_{\mathfrak{g}}\left(\frac{\sinh\frac{\mathrm{ad}_X}{2}}{\frac{\mathrm{ad}_X}{2}}\right)$ for S^1	$\det_{\mathfrak{g}}\left(\frac{\mathrm{ad}_X}{2}\cdot\coth\frac{\mathrm{ad}_X}{2}\right)^{-1}$ for Klein bottle

where $\varepsilon \in H^1$ – generator, $X \in \mathfrak{g}$ – variable.

Triangulation of the nilmanifold:



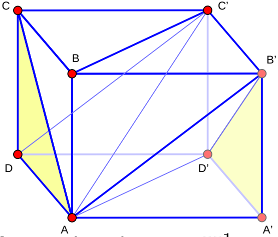
one **0-simplex:** A=B=C=D=A'=B'=C'=D'

seven **1-simplices:** AD=BC=A'D'=B'C', AA'=BB'=CC'=DD', AB=DC=D'B', AC=A'B'=D'C', AB'=DC', AD'=BC', AC'

twelve **2-simplices:** AA'B'=DD'C', AB'B=DC'C, AA'D'=BB'C', AD'D=BC'C, ACD=AB'D', ABC=D'B'C', AB'D', AC'D', ACC', ABC'

six **3-simplices:** AA'B'D', AB'C'D', ADC'D', ABB'C', ABCC', ACDC'

Triangulation of the nilmanifold:



one **0-simplex:** A=B=C=D=A'=B'=C'=D'

seven **1-simplices:** AD=BC=A'D'=B'C', AA'=BB'=CC'=DD', AB=DC=D'B', AC=A'B'=D'C', AB'=DC', AD'=BC', AC'

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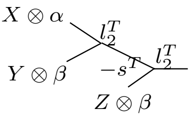
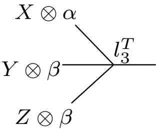
six **3-simplices:** AA'B'D', AB'C'D', ADC'D', ABB'C', ABCC', ACDC'

Massey bracket on H^1:

$$l_3(X\otimes[\alpha],Y\otimes[\beta],Z\otimes[\beta])=$$

$$=\frac{1}{2}$$


$$+\frac{1}{6}$$


+ permutations of inputs

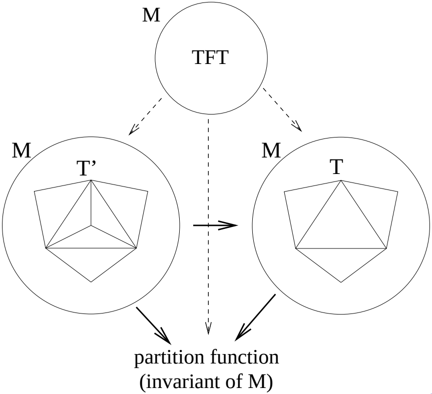
$$=([[X,Y],Z]+[[X,Z],Y])\otimes[\eta]\in\mathfrak{g}\otimes H^2(T)$$

where $s^T=d^\vee/(dd^\vee+d^\vee d)$;

$\alpha=e_{AC}+e_{AD}+e_{AC'}+e_{AD'}$, $\beta=e_{AA'}+e_{AB'}+e_{AC'}+e_{AD'}$

– representatives of cohomology classes $[\alpha]$, $[\beta]$ in simplicial cochains.

Simplicial program for TFTs: Given a TFT on a manifold M with space of fields F_M and action $S_M \in C^\infty(F_M)[[\hbar]]$, construct an **exact discretization** associating to a triangulation T of M a fin.dim. space F_T and a **local** action $S_T \in C^\infty(F_T)[[\hbar]]$, such that partition function Z_M and correlation functions can be obtained from (F_T, S_T) by fin.dim. integrals. Also, if T' is a subdivision of T, S_T is an effective action for $S_{T'}$.



Example of a TFT for which the exact discretization exists:
BF **theory:**

- fields: $F_M = \underbrace{\mathfrak{g} \otimes \Omega^1(M)}_{A} \oplus \underbrace{\mathfrak{g}^* \otimes \Omega^{\dim M - 2}(M)}_{B}$,
- action: $S_M = \int_M \langle B \overset{\wedge}{,} dA + A \wedge A \rangle$,
- equations of motion: $dA + A \wedge A = 0$, $d_A B = 0$.

Algebra – TFT dictionary

de Rham algebra $\mathfrak{g} \otimes \Omega^\bullet(M)$ (as a dg Lie algebra)	BF theory
unimodular L_∞ algebra $(V, \{l_n\}, \{q_n\})$	BF_∞ theory, $F = V[1] \oplus V^*[-2]$, $S = \sum_n \frac{1}{n!}\langle B, l_n(A, \cdots, A)\rangle +$ $+\hbar \sum_n \frac{1}{n!} q_n(A, \cdots, A)$
quadratic relations on operations	Batalin-Vilkoviski master equation $\underbrace{\Delta}_{\frac{\partial}{\partial A}\frac{\partial}{\partial B}} e^{S/\hbar} = 0$
homotopy transfer $V \to V'$	effective action $e^{S'/\hbar} = \int_{L \subset F''} e^{S/\hbar}$, $F = F' \oplus F''$
choice of chain homotopy s	gauge-fixing (choice of Lagrangian $L \subset F''$)

Batalin-Vilkovisky formalism

References: I. A. Batalin, G. A. Vilkovisky, *Gauge algebra and quantization*, Phys. Lett. B 102, 1 (1981) 27–31;
A. S. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. 155 2 (1993) 249–260.

Motivation: resolution of the problem of degenerate critical loci in perturbation theory ("gauge-fixing").

Definition

A *BV algebra* $(A, \cdot, \{,\}, \Delta)$ is a unital $\mathbb{Z}$-graded commutative algebra $(A^\bullet, \cdot, 1)$ endowed with:

- a degree 1 Poisson bracket $\{,\} : A \otimes A \to A$ — a bi-derivation of $\cdot$, satisfying Jacobi identity (i.e. $(A, \cdot, \{,\})$ is a *Gerstenhaber algebra*),
- a degree 1 operator ("BV Laplacian") $\Delta : A^\bullet \to A^{\bullet+1}$ satisfying

$$\Delta^2 = 0,\ \Delta(1) = 0,\ \Delta(a\cdot b) = (\Delta a)\cdot b+(-1)^{|a|}a\cdot(\Delta b)+(-1)^{|a|}\{a, b\}$$

Examples:

1. For $\mathcal{F}$ a $\mathbb{Z}$-graded manifold endowed with a degree -1 symplectic form ω and a "consistent" volume element μ (the data $(\mathcal{F},\omega,\mu)$ is called an "SP-manifold"), the ring of functions $A = C^\infty(\mathcal{F})$ carries a BV algebra structure, with pointwise multiplication $\cdot$, and with

$$\{f,g\} = \check{f}g, \quad \Delta f = \frac{1}{2}\mathrm{div}_\mu \check{f}$$

where $\check{f}$ is the Hamiltonian vector field for f defined by $\iota_{\check{f}}\omega = df$. Consistency condition on μ: $\Delta^2 = 0$.

2. Special case of the above when $(\mathcal{F},\omega)$ is a degree -1 symplectic graded vector space and μ is the translation-invariant volume element.

3. Polyvector fields on a manifold M carrying a volume element ρ, with opposite grading:

$$A^\bullet = \mathcal{V}^{-\bullet}(M), \quad \cdot = \wedge, \quad \{,\} = [,]_{NS}, \quad \Delta = \mathrm{div}_\rho$$

— this correspond to setting $\mathcal{F} = T^*[-1]M$ in (1).

Definition

Element $S \in A^0[[\hbar]]$ is said to satisfy Batalin-Vilkovisky quantum master equation (QME), if

$$\Delta e^{\frac{i}{\hbar}S} = 0$$

or equivalently in Maurer-Cartan form:

$$\frac{1}{2}\{S, S\} - i\hbar\Delta S = 0$$

Two solutions of QME, S and S' are said to be equivalent (related by a canonical transformation) if

$$e^{\frac{i}{\hbar}S'} = e^{\frac{i}{\hbar}S} + \Delta\left(e^{\frac{i}{\hbar}S}R\right)$$

for some generator $R \in A^{-1}[[\hbar]]$. For infinitesimal transformations:

$$S' = S + \{S, R\} - i\hbar\Delta R$$

Fix an SP-manifold $(\mathcal{F},\omega,\mu)$. Given a solution of QME $S\in C^\infty(\mathcal{F})[[\hbar]]$ and a Lagrangian submanifold $\mathcal{L}\subset\mathcal{F}$, one constructs the **BV integral**:

$$Z_{S,\mathcal{L}}=\int_{\mathcal{L}}e^{\frac{i}{\hbar}S}$$

BV-Stokes theorem (Batalin-Vilkovisky-Schwarz)

1. If $\mathcal{L},\mathcal{L}'\subset\mathcal{F}$ are two Lagrangian submanifolds that can be connected by a smooth family of Lagrangian submanifolds, then

$$Z_{S,\mathcal{L}}=Z_{S,\mathcal{L}'}$$

2. If S and S' are equivalent, then

$$Z_{S,\mathcal{L}}=Z_{S',\mathcal{L}}$$

Let $(\mathcal{F} = \mathcal{F}' \times \mathcal{F}'', \omega = \omega' + \omega'', \mu = \mu' \times \mu'')$ be a product of two SP-manifolds and S a solution of QME on $\mathcal{F}$. Define the effective BV action S' on $\mathcal{F}'$ by the **fiberwise BV integral**

$$e^{\frac{i}{\hbar}S'} = \int_{\mathcal{L}'' \subset \mathcal{F}''} e^{\frac{i}{\hbar}S}$$

where $\mathcal{L}''$ is a Lagrangian submanifold of $\mathcal{F}''$.

Theorem (P.M.)

1. Effective BV action S' satisfies QME on $\mathcal{F}'$.
2. If $\mathcal{L}'', \tilde{\mathcal{L}}''$ are two Lagrangian submanifolds of $\mathcal{F}''$ that can be connected by a smooth family of Lagrangian submanifolds, then corresponding effective actions are equivalent.
3. If S, $\tilde{S}$ are two equivalent solutions of QME on $\mathcal{F}$, then the corresponding effective actions on $\mathcal{F}'$ are equivalent.

Thus the effective BV action construction defines the push-forward

$$\begin{array}{c}(\text{solutions of QME on } \mathcal{F})/\text{equivalence} \\ \Big\downarrow \text{fiberwise BV integral} \\ (\text{solutions of QME on } \mathcal{F}')/\text{equivalence}\end{array}$$

One-dimensional simplicial Chern-Simons theory

Reference: A. Alekseev, P. Mnev, *One-dimensional Chern-Simons theory*, Comm. in Math. Phys. 307 1 (2011) 185–227

Continuum theory on a circle. Fix $(\mathfrak{g}, \langle,\rangle)$ be a *quadratic* even-dimensional Lie algebra.

- Space of fields: $\mathcal{F} = \underbrace{\Pi\mathfrak{g}\otimes\Omega^0(S^1)}_{\psi} \oplus \underbrace{\mathfrak{g}\otimes\Omega^1(S^1)}_{A}$ – a $\mathbb{Z}_2$-graded manifold with an odd symplectic structure coming from Poincaré duality on S^1: $\omega = \int_{S^1}\langle\delta\psi \overset{\wedge}{,} \delta A\rangle$
- Action: $S(\psi, A) = \int_{S^1}\langle\psi \overset{\wedge}{,} d\psi + [A,\psi]\rangle$

Effective BV action on cochains of triangulated circle.
Denote T_N the triangulation of S^1 with N vertices. Discrete space of fields:

$$\mathcal{F}_{T_N} = \Pi\mathfrak{g}\otimes C^0(T_N)\oplus\mathfrak{g}\otimes C^1(T_N)$$

with coordinates $\{\psi_k\in\Pi\mathfrak{g}, A_k\in\mathfrak{g}\}_{k=1}^N$ and odd symplectic form

$$\omega_{T_N} = \sum_{k=1}^N \left\langle \delta\underbrace{\left(\frac{\psi_k+\psi_{k+1}}{2}\right)}_{\tilde{\psi}_k}, \delta A_k\right\rangle$$

Explicit simplicial Chern-Simons action on cochains of triangulated circle:

$$S_{T_N} =$$
$$= -\frac{1}{2}\sum_{k=1}^{N}\left((\psi_k,\psi_{k+1}) + \frac{1}{3}(\psi_k, \mathrm{ad}_{A_k}\psi_k) + \frac{1}{3}(\psi_{k+1}, \mathrm{ad}_{A_k}\psi_{k+1}) + \frac{1}{3}(\psi_k, \mathrm{ad}_{A_k}\psi_{k+1})\right) +$$
$$+\frac{1}{2}\sum_{k=1}^{N}(\psi_{k+1}-\psi_k, \left(\frac{1-R(\mathrm{ad}_{A_k})}{2}\left(\frac{1}{1+\mu_k(A')} - \frac{1}{1+R(\mathrm{ad}_{A_k})}\right)\frac{1-R(\mathrm{ad}_{A_k})}{2R(\mathrm{ad}_{A_k})} +\right.$$
$$\left. +(\mathrm{ad}_{A_k})^{-1} + \frac{1}{12}\mathrm{ad}_{A_k} - \frac{1}{2}\coth\frac{\mathrm{ad}_{A_k}}{2}\right)\circ(\psi_{k+1}-\psi_k))+$$
$$+\frac{1}{2}\sum_{k'=1}^{N}\sum_{k=k'+1}^{k'+N-1}(-1)^{k-k'}(\psi_{k+1}-\psi_k, \frac{1-R(\mathrm{ad}_{A_k})}{2}R(\mathrm{ad}_{A_{k-1}})\cdots R(\mathrm{ad}_{A_{k'}})\cdot$$
$$\cdot\frac{1}{1+\mu_{k'}(A')}\cdot\frac{1-R(\mathrm{ad}_{A_{k'}})}{2R(\mathrm{ad}_{A_{k'}})}\circ(\psi_{k'+1}-\psi_{k'}))+$$
$$+\hbar\,\frac{1}{2}\,\mathrm{tr}_{\mathfrak{g}}\log\left((1+\mu_\bullet(A'))\prod_{k=1}^{n}\left(\frac{1}{1+R(\mathrm{ad}_{A_k})}\cdot\frac{\sinh\frac{\mathrm{ad}_{A_k}}{2}}{\frac{\mathrm{ad}_{A_k}}{2}}\right)\right)$$

where

$$R(\mathcal{A}) = -\frac{\mathcal{A}^{-1}+\frac{1}{2}-\frac{1}{2}\coth\frac{\mathcal{A}}{2}}{\mathcal{A}^{-1}-\frac{1}{2}-\frac{1}{2}\coth\frac{\mathcal{A}}{2}},\quad \mu_k(A') = R(\mathrm{ad}_{A_{k-1}})R(\mathrm{ad}_{A_{k-2}})\cdots R(\mathrm{ad}_{A_{k+1}})R(\mathrm{ad}_{A_k})$$

Questions:

- Why such a long formula?
- It is not simplicially local (there are monomials involving distant simplices). How to disassemble the result into contributions of individual simplices?
- How to check quantum master equation for S_{T_N} explicitly?
- Simplicial aggregations should be given by finite-dimensional BV integrals; how to check that?

1D simplicial Chern-Simons as Atiyah's TFT

Set

$$\zeta(\underbrace{\tilde{\psi}}_{\in \Pi\mathfrak{g}}, \underbrace{A}_{\in \mathfrak{g}}) = (i\hbar)^{-\frac{\dim \mathfrak{g}}{2}} \int_{\Pi\mathfrak{g}} D\lambda \, \exp\left(-\frac{1}{2\hbar}\langle\hat{\psi},[A,\hat{\psi}]\rangle + \langle\lambda,\hat{\psi}-\tilde{\psi}\rangle\right) \in Cl(\mathfrak{g})$$

where $\{\hat{\psi}^a\}$ are generators of the Clifford algebra $Cl(\mathfrak{g})$, $\hat{\psi}^a\hat{\psi}^b + \hat{\psi}^b\hat{\psi}^a = \hbar\delta^{ab}$

Element ζ can be used as a **building block** (partition function for an interval with standard triangulation) for 1D Chern-Simons as Atiyah's TFT on triangulated 1-cobordisms Θ, with

- Partition functions
  $Z_\Theta \in C^\infty(\underbrace{\Pi\mathfrak{g}\otimes C_1(\Theta)\oplus\mathfrak{g}\otimes C^1(\Theta)}_{\mathcal{F}_\Theta}) \otimes Cl(\mathfrak{g})^{\otimes\,\#\{\text{intervals}\}}$,
- For a disjoint union, $Z_{\Theta_1\sqcup\Theta_2} = Z_{\Theta_1}\otimes Z_{\Theta_2}$,
- For a concatenation of two triangulated intervals, $Z_{\Theta_1\cup\Theta_2} = Z_{\Theta_1} * Z_{\Theta_2}$ – Clifford product,
- For the closure of a triangulated interval Θ into a triangulated circle Θ', $Z_{\Theta'} = \mathrm{Str}_{Cl(\mathfrak{g})} Z_\Theta$ – Clifford supertrace.

Theorem (A.Alekseev, P.M.)

1. For a triangulated circle,
$$Z_{T_N} = \mathrm{Str}_{Cl(\mathfrak{g})}\left(\zeta(\tilde{\psi}_N, A_N) * \cdots * \zeta(\tilde{\psi}_1, A_1)\right) = e^{\frac{i}{\hbar}S_{T_N}}$$
2. For a triangulated interval, the partition function satisfies the *modified* quantum master equation
$$\hbar\Delta_\Theta Z_\Theta + \frac{1}{\hbar}\left[\frac{1}{6}\langle\hat{\psi}, [\hat{\psi}, \hat{\psi}]\rangle, Z_\Theta\right]_{Cl(\mathfrak{g})} = 0$$
where $\Delta_\Theta = \sum_k \frac{\partial}{\partial\tilde{\psi}_k}\frac{\partial}{\partial A_k}$.
3. Simplicial action on triangulated circle S_{T_N} satisfies the usual BV quantum master equation, $\Delta_{T_N} e^{\frac{i}{\hbar}S_{T_N}} = 0$.

The space of states for a point. Fix a complex polarization $\mathfrak{g}\otimes\mathbb{C} = \mathfrak{h}\oplus\bar{\mathfrak{h}}$. Then one has an isomorphism $\rho : Cl(\mathfrak{g}) \to C^\infty(\Pi\mathfrak{h})\otimes C^\infty(\Pi\bar{\mathfrak{h}})$ Thus we set

$$\mathcal{H}_{pt+} = C^\infty(\Pi\mathfrak{h}), \quad \mathcal{H}_{pt-} = C^\infty(\Pi\bar{\mathfrak{h}}) \simeq (\mathcal{H}_{pt+})^*$$

The building block ζ can be written as a path integral with boundary conditions:

$$\rho(\zeta)(\underbrace{\eta_{\mathrm{out}}}_{\in\Pi\mathfrak{h}},\underbrace{\bar{\eta}_{\mathrm{in}}}_{\in\Pi\bar{\mathfrak{h}}};\tilde{\psi},A) = \int_{\substack{\pi\psi(1)=\eta_{\mathrm{out}},\\ \bar{\pi}\psi(0)=\bar{\eta}_{\mathrm{in}},\\ \int_0^1 dt\,\psi=\tilde{\psi}}} \mathcal{D}\psi \quad e^{\frac{i}{\hbar}\int_0^1\langle\psi\,\overset{\wedge}{,}\,d\psi+[Adt,\psi]\rangle}$$

where $\pi:\mathfrak{g}_\mathbb{C}\to\mathfrak{h}$, $\bar{\pi}:\mathfrak{g}_\mathbb{C}\to\bar{\mathfrak{h}}$ are the projections to the two terms in $\mathfrak{g}_\mathbb{C}\simeq\mathfrak{h}\oplus\bar{\mathfrak{h}}$.

Classical BV structure for gauge theory on a closed manifold:
A graded manifold $\mathcal{F}$ (space of fields) endowed with

- a cohomological vector field Q of degree 1, $Q^2 = 0$,
- a degree -1 symplectic form ω,
- a degree 0 Hamiltonian function S generating the cohomological vector field: $\delta S = \iota_Q \omega$

Extension to manifolds with boundary ("BV-BFV formalism").
To a manifold Σ with boundary $\partial\Sigma$ a gauge theory associates:

- **Boundary BFV data:** a graded manifold $\mathcal{F}_\partial$ endowed with
  - a degree 1 cohomological vector field Q_∂,
  - a degree 0 **exact** symplectic form $\omega_\partial = \delta\alpha_\partial$,
  - a degree 1 Hamiltonian S_∂ generating Q_∂, i.e. $Q_\partial = \{S_\partial, \bullet\}_{\omega_\partial}$.
- **Bulk BV data:** a graded manifold $\mathcal{F}$ endowed with
  - a degree 1 cohomological vector field Q,
  - a projection $\pi : \mathcal{F} \to \mathcal{F}_\partial$ which is a Q-morphism, i.e. $d\pi(Q) = Q_\partial$,
  - a degree -1 symplectic form ω,
  - a degree 0 function S satisfying $\delta S = \iota_Q \omega + \pi^* \alpha_\partial$.

Reference: A. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary*, arXiv:1201.0290

Euler-Lagrange spaces.
One can define coisotropic submanifolds $\mathcal{EL} \subset \mathcal{F}$, $\mathcal{EL}_\partial \subset \mathcal{F}_\partial$ as zero loci of Q and Q_∂ respectively. For "nice" theories, the "evolution relation" $\mathcal{L} = \pi(\mathcal{EL}) \subset \mathcal{EL}_\partial \subset \mathcal{F}_\partial$ is **Lagrangian**.
Reduction: EL moduli spaces.
One can quotient Euler-Lagrange spaces by the distribution induced from the cohomological vector field to produce *EL moduli spaces* $\mathcal{M} = \mathcal{EL}/Q$, $\mathcal{M}_\partial = \mathcal{EL}/Q_\partial$. They carry the following structure induced from BV-BFV structure on fields:

- map $\pi_* : \mathcal{M} \to \mathcal{M}_\partial$,
- $\mathcal{M}_\partial$ is degree 0 symplectic, $\mathcal{M}$ is degree 1 Poisson,
- image of π_* is Lagrangian, fibers of π_* comprise the symplectic foliation of $\mathcal{M}$,
- a line bundle L over $\mathcal{M}_\partial$ with connection ∇ of curvature being the symplectic form on $\mathcal{M}_\partial$,
- a horizontal section of the pull-back bundle $(\pi_*)^*L$.

A simple example: abelian Chern-Simons theory on a 3-manifold Σ with boundary.

$$\mathcal{F} = \Omega^\bullet(\Sigma), \quad S = \frac{1}{2}\int_\Sigma A \wedge dA, \quad \omega = \frac{1}{2}\int_\Sigma \delta A \wedge \delta A,$$

$$\mathcal{F}_\partial = \Omega^\bullet(\partial\Sigma), \quad S_\partial = \frac{1}{2}\int_{\partial\Sigma} A_\partial \wedge dA_\partial, \quad \alpha_\partial = \frac{1}{2}\int_{\partial\Sigma} A_\partial \wedge \delta A_\partial$$

Euler-Lagrange spaces: $\mathcal{EL} = \Omega^\bullet_{\text{closed}}(\Sigma)$, $\mathcal{EL}_\partial = \Omega^\bullet_{\text{closed}}(\partial\Sigma)$.
EL moduli spaces: $\mathcal{M} = H^\bullet(\Sigma)$, $\mathcal{M}_\partial = H^\bullet(\partial\Sigma)$.
Non-abelian Chern-Simons theory. EL moduli spaces are (derived versions of) the moduli spaces of flat G-bundles over Σ and $\partial\Sigma$.

Remarks:

- One can introduce the third EL moduli space $\mathcal{M}_{\text{rel}}$, so that the triple $(\mathcal{M}_{\text{rel}}, \mathcal{M}, \mathcal{M}_\partial)$ supports long exact sequence for tangent spaces, Lefschetz duality, Meyer-Vietoris type gluing.
- EL moduli spaces come with a cohomological description, $\mathcal{M} = \operatorname{Spec} H_Q(C^\infty(\mathcal{F}))$ which is particularly useful for quantization. (E.g. we get a simple cohomological description of **Verlinde space**, arising as the geometric quantization of the moduli space of local systems).

Idea of quantization.
Take a foliation of $\mathcal{F}_\partial$ by Lagrangian submanifolds. Each leaf of the foliation is a valid boundary condition for bulk fields in the path integral. Space of states is constructed as

$$\mathcal{H}_{\partial\Sigma} = \mathrm{Fun}\{\text{space of leaves of the foliation}\}$$

with a differential $\hat{S}_\partial$. Partition function, constructed by the path integral, is a function of the leaf and of the bulk zero-modes (i.e. function on fiber of $\pi_* : \mathcal{M} \to \mathcal{M}_\partial$), and is expected to satisfy a version of quantum master equation:

$$(\Delta_{\mathrm{bulk\,z.m.}} + \hat{S}_\partial) Z_\Sigma = 0$$

Developments

- Axelrod-Singer's perturbative treatment of Chern-Simons on closed manifolds extended to **non-acyclic** background flat connections. Algebraic model of Chern-Simons based on dg Frobenius algebras studied.
  Reference: A. Cattaneo, P. Mnev, *Remarks on Chern-Simons invariants*, Comm. in Math. Phys. 293 3 (2010) 803-836
- Global perturbation theory for Poisson sigma model studied from the standpoint of formal geometry of the target. Genus 1 partition function with Kähler target is shown yield Euler characteristic of the target.
  Reference: F. Bonechi, A. Cattaneo, P. Mnev, *The Poisson sigma model on closed surfaces*, JHEP 99 1 (2012) 1-27
- A class of generalized Wilson loop observables constructed via BV push-forward of the transgression of a Hamiltonian Q-bundle over the target to the mapping space.
  Reference: P. Mnev, *A construction of observables for AKSZ sigma models*, arXiv:1212.5751 (math-ph)
- Cohomology of $\hat{S}_\partial$ on the canonical quantization of boundary BFV phase space of Chern-Simons with Wilson lines yields the space of conformal blocks of Wess-Zumino-Witten model.
  Reference: A. Alekseev, Y. Barmaz, P. Mnev, *Chern-Simons theory with Wilson lines and boundary in the BV-BFV formalism*, J.Geom. and Phys. 67 (2013) 1-15

Program

- Construct perturbative quantization of TFTs in the BV-BFV formalism as a (far-reaching) extension of Axelrod-Singer's construction. Possible application: link between Reshetikhin-Turaev invariant and Chern-Simons theory.
- Study applications to invariants of manifolds and knots consistent with surgery. (In particular, study the extension of gluing formulae for cohomology and Ray-Singer torsion to higher perturbative invariants, e.g. Axelrod-Singer and Bott-Cattaneo invariants of 3-manifolds.)
- Further study of EL moduli spaces (and their geometric quantization) from the point of view of derived symplectic geometry.
- Extend the construction to allow manifolds with corners; compare the results with Baez-Dolan-Lurie axioms for extended TFTs.