Towards perturbative topological field theory on manifolds with boundary

Pavel Mnev

University of Zurich

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Plan of the talk

- Background: topological field theory
- Hidden algebraic structure on cohomology of simplicial complexes coming from TFT
- One-dimensional simplicial Chern-Simons theory
- Topological field theory on manifolds with boundary
Axioms of an \( n \)-dimensional topological quantum field theory. (Atiyah’88)

**Data:**

1. To a closed \((n - 1)\)-dimensional manifold \( B \) a TFT associates a vector space \( \mathcal{H}_B \) (the “space of states”).

2. To a \( n \)-dimensional cobordism \( \Sigma : B_1 \rightarrow B_2 \) a TFT associates a linear map \( Z_\Sigma : \mathcal{H}_{B_1} \rightarrow \mathcal{H}_{B_2} \) (the “partition function”).

3. Representation of \( \text{Diff}(B) \) on \( \mathcal{H}_B \).
Axioms:

(a) Multiplicativity “⊔ → ⊗”:

\[ \mathcal{H}_{B_1 \sqcup B_2} = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}, \quad Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2} \]

(b) Gluing axiom: for cobordisms \( \Sigma_1 : B_1 \to B_2, \Sigma_2 : B_2 \to B_3, \)

\[ Z_{\Sigma_1 \sqcup B_2 \Sigma_2} = Z_{\Sigma_2} \circ Z_{\Sigma_1} \]

(c) Normalization: \( \mathcal{H}_{\emptyset} = \mathbb{C}. \)

(d) Diffeomorphisms of \( \Sigma \) constant on \( \partial \Sigma \) do not change \( Z_{\Sigma} \). Under general diffeomorphisms, \( Z_{\Sigma} \) transforms equivariantly.

Remarks:

- A closed \( n \)-manifold \( \Sigma \) can be viewed as a cobordism \( \emptyset \xrightarrow{\Sigma} \emptyset \), so \( Z_{\Sigma} : \mathbb{C} \to \mathbb{C} \) is a multiplication by a complex number – a diffeomorphism invariant of \( \Sigma \).

- An \( n \)-TFT \( (\mathcal{H}, Z) \) is a functor of symmetric monoidal categories \( \text{Cob}_n \to \text{Vect}_\mathbb{C} \), with diffeomorphisms acting by natural transformations.

A. S. Schwarz’78: path integral of the form

\[ Z_\Sigma = \int_{F_\Sigma} D X \ e^{i \hbar S(X)} \]

with \( S \) a local functional on \( F_\Sigma \) (a space of sections of a sheaf over \( \Sigma \)), invariant under \( \text{Diff}(\Sigma) \), can produce a topological invariant of \( \Sigma \) (when it can be defined, e.g. through formal stationary phase expression at \( \hbar \to 0 \)).

**Example:** Let \( \Sigma \) be odd-dimensional, closed, oriented; let \( E \) be an acyclic local system, \( F_\Sigma = \Omega^r(\Sigma, E) \oplus \Omega^{\text{dim } \Sigma - r - 1}(\Sigma, E^*) \) with \( 0 \leq r \leq \text{dim } \Sigma - 1 \), and with the action

\[ S = \int_\Sigma \langle b \wedge da \rangle \]

The corresponding path integral can be defined and yields the **Ray-Singer torsion** of \( \Sigma \) with coefficients in \( E \).

Witten’89: Let $\Sigma$ be a compact, oriented, framed 3-manifold, $G$ – a compact Lie group, $P = \Sigma \times G$ the trivial $G$-bundle over $\Sigma$. Set $F_\Sigma = \text{Conn}(P) \simeq \mathfrak{g} \otimes \Omega^1(\Sigma)$ – the space of connections in $P$; $\mathfrak{g} = \text{Lie}(G)$. For $A$ a connection, set

$$S_{CS}(A) = \text{tr}_\mathfrak{g} \int_\Sigma \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A$$

– the integral of the Chern-Simons 3-form. Consider

$$Z_\Sigma(k) = \int_{\text{Conn}(P)} DA \ e^{\frac{ik}{\hbar} S_{CS}(A)}$$

for $k = 1, 2, 3, \ldots$ (i.e. $\hbar = \frac{2\pi}{k}$). For closed manifolds, $Z(\Sigma, k)$ is an interesting invariant, calculable explicitly through surgery. E.g. for $G = SU(2), \Sigma = S^3$, the result is

$$Z_{S^3}(k) = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi}{k+2} \right)$$
The space of states $\mathcal{H}_B$ corresponding to a surface $B$ is the geometric quantization of the moduli space of local systems $\text{Hom}(\pi_1(B), G)/G$ with Atiyah-Bott symplectic structure.

For a knot $\gamma : S^1 \hookrightarrow \Sigma$, Witten considers the expectation value

$$W(\Sigma, \gamma, k) = Z_{\Sigma}(k)^{-1} \int_{\text{Conn}(P)} D A \, e^{\frac{ik}{2\pi} \text{CS}(A)} \, \text{tr}_R \, \text{hol}(\gamma^* A)$$

where $R$ is a representation of $G$. In case $G = SU(2)$, $\Sigma = S^3$, this expectation value yields the value of Jones’ polynomial of the knot at the point $q = e^{\frac{i\pi}{k+2}}$.

### Background: Lagrangian TFTs

**Axelrod-Singer'94**: Perturbation theory (formal stationary phase expansion at \( \hbar \to 0 \)) for Chern-Simons theory on a **closed**, oriented, framed 3-manifold rigorously constructed.
\[ Z_{\Sigma}^{\text{pert}}(A_0, \hbar) = e^{i\frac{\hbar}{\pi} S_{CS}(A_0)} \tau(\Sigma, A_0) e^{i\pi \frac{\eta(\Sigma, A_0, g)}{2}} e^{ic(\hbar) S_{\text{grav}}(g)} \]

\[ \cdot \exp \left( \frac{i}{\hbar} \sum_{\text{connected 3-valent graphs } \Gamma} \frac{(i\hbar)^{l(\Gamma)}}{|\text{Aut}(\Gamma)|} \int_{\text{Conf}_V(\Gamma)(\Sigma)} \prod_{\text{edges}} \pi_{e_1 e_2} \eta \right) \]

where
- \( A_0 \) is a fixed \textbf{acyclic} flat connection, \( g \) is an arbitrary Riemannian metric,
- \( \tau(\Sigma, A_0) \) is the Ray-Singer torsion, \( \eta(\Sigma, A_0, g) \) is the Atiyah’s eta-invariant,
- \( V(\Gamma) \) and \( l(\Gamma) \) are the number of vertices and the number of loops of a graph,
- \( \text{Conf}_n(\Sigma) \) is the Fulton-Macpherson-Axelrod-Singer compactification of the configuration space of \( n \)-tuple distinct points on \( \Sigma \),
- \( \eta \in \Omega^2(\text{Conf}_2(\Sigma)) \) is the \textbf{propagator}, a parametrics for the Hodge-theoretic inverse of de Rham operator, \( d/(dd^* + d^*d) \),
- \( \pi_{ij} : \text{Conf}_n(\Sigma) \to \text{Conf}_2(\Sigma) \) – forgetting all points except \( i \)-th and \( j \)-th.
- \( S_{\text{grav}}(g) \) is the Chern-Simons action evaluated on the Levi-Civita connection, \( c(\hbar) \in \mathbb{C}[[\hbar]] \).
Remarks:

- Expression for $\log Z$ is finite in each order in $\hbar$: given as a finite sum of integrals of smooth forms over compact manifolds.
- Propagator depends on the choice of metric $g$, but the whole expression does not depend on $g$.

Comments:

- Explicit examples of Atiyah’s 3-TFTs were constructed by Reshetikhin-Turaev’91 and Turaev-Viro’92 from representation theory of quantum groups at roots of unity.

- Main motivation to study TFTs is that they produce invariants of manifolds and knots.

- Example of a different application: use of the 2-dimensional Poisson sigma model on a disc in Kontsevich’s deformation quantization of Poisson manifolds (Kontsevich'97, Cattaneo-Felder'00).
Problems:

1. Witten’s treatment of Chern-Simons theory is not completely mathematically transparent (use of path integral as a “black box” which is assumed to have certain properties); Axelrod-Singer’s treatment is transparent, but restricted to closed manifolds: perturbative Chern-Simons theory as Atiyah’s TFT is not yet constructed.

2. Reshetikhin-Turaev invariants are conjectured to coincide asymptotically with the Chern-Simons partition function.

3. Construct a combinatorial model of Chern-Simons theory on triangulated manifolds, retaining the properties of a perturbative gauge theory and yielding the same manifold invariants.
Program/logic of the exposition:

Simplicial $BF$ theory \textbf{(P.M.)}

\[ \rightarrow \text{hidden algebraic structure on cohomology of simplicial complexes} \]

\[ \downarrow \]

One-dimensional simplicial Chern-Simons theory

\[ \text{(with A. Alekseev)} \]

\[ \downarrow \]

Perturbative TFT on manifolds with boundary

\[ \rightarrow \text{Euler-Lagrange moduli spaces: supergeometric structures, gluing, cohomological quantization. Gluing formulae for quantum invariants.} \]

\[ \text{(partially complete, with A. Cattaneo and N. Reshetikhin)} \]

\[ \downarrow \]

Perturbative TFT on manifolds with corners \textbf{(in progress)}
Background: simplicial complexes, cohomological operations

Simplicial complex $T$
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Simplicial complex $T$

\[
\begin{align*}
\text{Simplicial cochains } C^0(T) & \rightarrow \cdots \rightarrow C^\text{top}(T), \\
C^k(T) &= \text{Span}\{k - \text{simplices}\}, \\
d_k : C^k(T) &\rightarrow C^{k+1}(T), \\
\sum_{\sigma' \in T: \sigma \in \text{faces}(\sigma')} &\pm e_{\sigma'}
\end{align*}
\]
Simplicial complex $T$

\[ \xrightarrow{} \]

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\[ d_k : C^k(T) \to C^{k+1}(T), \quad e_\sigma \mapsto \sum_{\sigma' \in T : \sigma \in \text{faces}(\sigma')} \pm e_{\sigma'} \]

Cohomology $H^\bullet(T)$, $H^k(T) = \ker d_k / \text{im } d_{k-1}$

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a homotopy invariant of $T$
Cohomology carries a commutative ring structure, coming from (non-commutative) Alexander’s product for cochains.
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Massey operations on cohomology are a complete invariant of rational homotopy type in simply connected case (Quillen-Sullivan), i.e. rationalized homotopy groups $\mathbb{Q} \otimes \pi_k(T)$ can be recovered from them.
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Massey operations on cohomology are a complete invariant of rational homotopy type in simply connected case (Quillen-Sullivan), i.e. rationalized homotopy groups $\mathbb{Q} \otimes \pi_k(T)$ can be recovered from them.

Example of use: linking of Borromean rings is detected by a non-vanishing Massey operation on cohomology of the complement.\[m_3([\alpha], [\beta], [\gamma]) = [u \wedge \gamma + \alpha \wedge v] \in H^2\]
where $[\alpha], [\beta], [\gamma] \in H^1, du = \alpha \wedge \beta, dv = \beta \wedge \gamma.$
Another example: \textbf{nilmanifold}

\[ M = H_3(\mathbb{R})/H_3(\mathbb{Z}) \]

\[
= \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} / \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}
\]

Denote

\[ \alpha = dx, \ \beta = dy, \ u = dz - ydx \in \Omega^1(M) \]

Important point: \( \alpha \wedge \beta = du \). The cohomology is spanned by classes

\[
\begin{align*}
\{[1]\} & \quad \text{degree 0} \\
\{[\alpha], [\beta]\} & \quad \text{degree 1} \\
\{[\alpha \wedge u], [\beta \wedge u]\} & \quad \text{degree 2} \\
\{[\alpha \wedge \beta \wedge u]\} & \quad \text{degree 3}
\end{align*}
\]

and

\[ m_3([\alpha], [\beta], [\beta]) = [u \wedge \beta] \in H^2(M) \]

is a non-trivial Massey operation.
Fix $\mathfrak{g}$ a unimodular Lie algebra (i.e. with $\text{tr}[x, \bullet] = 0$ for any $x \in \mathfrak{g}$).

Main construction (P.M.)

Simplicial complex $T$

\[\text{local formula}\]

Unimodular $L_\infty$ algebra structure on $\mathfrak{g} \otimes C^\bullet(T)$

\[\text{homotopy transfer}\]

Unimodular $L_\infty$ algebra structure on $\mathfrak{g} \otimes H^\bullet(T)$
Fix \( g \) a unimodular Lie algebra (i.e. with \( \text{tr}[x, \bullet] = 0 \) for any \( x \in g \)).

**Main construction (P.M.)**

Simplicial complex \( T \)

\[
\begin{align*}
\text{local formula} & \downarrow \\
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\text{homotopy transfer} & \\
\text{Unimodular } L_\infty \text{ algebra structure on } g \otimes H^\bullet(T)
\end{align*}
\]

**Main theorem (P.M.)**

Unimodular \( L_\infty \) algebra structure on \( g \otimes H^\bullet(T) \) (up to isomorphisms) is an invariant of \( T \) under simple homotopy equivalence.
Main construction (P.M.)

Simplicial complex $T$

$\downarrow$ local formula

Unimodular $L_\infty$ algebra structure on $g \otimes C^\bullet(T)$

$\downarrow$ homotopy transfer

Unimodular $L_\infty$ algebra structure on $g \otimes H^\bullet(T)$

- Thom’s problem: lifting ring structure on $H^\bullet(T)$ to a **commutative** product on cochains. Removing $g$, we get a homotopy commutative algebra on $C^\bullet(T)$. This is an improvement of Sullivan’s result with cDGA structure on cochains $= \Omega_{\text{poly}}(T)$.
- **Local** formulae for Massey operations.
- Our invariant is strictly stronger than rational homotopy type.
References:

A unimodular $L_\infty$ algebra is the following collection of data:

(a) a $\mathbb{Z}$-graded vector space $V^\bullet$,
(b) “classical operations” $l_n : \wedge^n V \to V$, $n \geq 1$,
(c) “quantum operations” $q_n : \wedge^n V \to \mathbb{R}$, $n \geq 1$, 

Note: First classical operation satisfies $(l_1)^2 = 0$, so $(V^\bullet, l_1)$ is a complex.

A unimodular $L_\infty$ algebra is in particular an $L_\infty$ algebra (as introduced by Lada-Stasheff), by ignoring $q_n$. Unimodular Lie algebra is the same as unimodular $L_\infty$ algebra with $l_1 = q_1 = 0$. 
Definition

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subject to two sequences of quadratic relations:

1. $\sum_{r+s=n} \frac{1}{r! s!} l_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$, $n \geq 1$
   (anti-symmetrization over inputs implied),

2. $\frac{1}{n!} \text{Str} l_{n+1}(\bullet, \cdots, \bullet, -) + $ $\sum_{r+s=n} \frac{1}{r! s!} q_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$
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- First classical operation satisfies $(l_1)^2 = 0$, so $(V^\bullet, l_1)$ is a complex.
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- Unimodular Lie algebra is the same as unimodular $L_\infty$ algebra with $l\neq 2 = q_\bullet = 0$. 
An alternative definition

A unimodular $L_\infty$ algebra is a graded vector space $V$ endowed with

- a vector field $Q$ on $V[1]$ of degree 1,
- a function $\rho$ on $V[1]$ of degree 0,

satisfying the following identities:

$$[Q, Q] = 0, \quad \text{div } Q = Q(\rho)$$
Homotopy transfer theorem (P.M.)

If \((V, \{l_n\}, \{q_n\})\) is a unimodular \(L_\infty\) algebra and \(V' \hookrightarrow V\) is a deformation retract of \((V, l_1)\), then

1. \(V'\) carries a unimodular \(L_\infty\) structure given by

\[
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where \(\Gamma_0\) runs over rooted trees with \(n\) leaves and \(\Gamma_1\) runs over 1-loop graphs with \(n\) leaves.
Homotopy transfer theorem (P.M.)

If \((V, \{l_n\}, \{q_n\})\) is a unimodular \(L_\infty\) algebra and \(V' \hookrightarrow V\) is a deformation retract of \((V, l_1)\), then

\[ V' \text{ carries a unimodular } L_\infty \text{ structure given by} \]

\[ l'_n = \sum \Gamma_0 \frac{1}{|\text{Aut}(\Gamma_0)|} : \wedge^n V' \rightarrow V' \]

\[ q'_n = \sum \Gamma_1 \frac{1}{|\text{Aut}(\Gamma_1)|} + \sum \Gamma_0 \frac{1}{|\text{Aut}(\Gamma_0)|} : \wedge^n V' \rightarrow \mathbb{R} \]

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<tbody>
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<td>edge</td>
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<td>((m + 1))-valent vertex</td>
<td>(l_m)</td>
</tr>
<tr>
<td>cycle</td>
<td>super-trace over (V)</td>
<td>(m)-valent (\circ)-vertex</td>
<td>(q_m)</td>
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where \(s\) is a chain homotopy, \(l_1 s + s l_1 = \text{id} - i p\).
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where \(s\) is a chain homotopy, \(l_1 s + s l_1 = \text{id} - i p\).

2. Algebra \((V', \{l'_n\}, \{q'_n\})\) changes by isomorphisms under changes of induction data \((i, p, s)\).
Introduction

**Locality of the algebraic structure on simplicial cochains**

\[ l_n^T(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n}) = \sum_{\sigma \in T : \sigma_1, \ldots, \sigma_n \in \text{faces}(\sigma)} \bar{l}_n^\sigma(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n})e_{\sigma} \]

\[ q_n^T(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n}) = \sum_{\sigma \in T : \sigma_1, \ldots, \sigma_n \in \text{faces}(\sigma)} \bar{q}_n^\sigma(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n}) \]

**Notations:** \( e_{\sigma} \) – basis cochain for a simplex \( \sigma \), \( X_{\bullet} \in g \), \( Xe_{\sigma} := X \otimes e_{\sigma} \).

Here \( \bar{l}_n^\sigma : \wedge^n(g \otimes C^\bullet(\sigma)) \to g \), \( \bar{q}_n^\sigma : \wedge^n(g \otimes C^\bullet(\sigma)) \to \mathbb{R} \) are universal local building blocks, depending on dimension of \( \sigma \) only, not on combinatorics of \( T \).
Zero-dimensional simplex $\sigma = [A]$

$\bar{l}_2(X e_A, Y e_A) = [X, Y]$, all other operations vanish.
**Zero-dimensional simplex** $\sigma = [A]$: 
$\bar{l}_2(X e_A, Y e_A) = [X, Y]$, all other operations vanish.

**One-dimensional simplex** $\sigma = [AB]$:

\[
\bar{l}_{n+1}(X_1 e_{AB}, \cdots, X_n e_{AB}, Y e_B) = \frac{B_n}{n!} \sum_{\theta \in S_n} [X_{\theta_1}, \cdots, [X_{\theta_n}, Y] \cdots]
\]

\[
\bar{l}_{n+1}(X_1 e_{AB}, \cdots, X_n e_{AB}, Y e_A) = (-1)^{n+1} \frac{B_n}{n!} \sum_{\theta \in S_n} [X_{\theta_1}, \cdots, [X_{\theta_n}, Y] \cdots]
\]

\[
\bar{q}_n(X_1 e_{AB}, \cdots, X_n e_{AB}) = \frac{B_n}{n \cdot n!} \sum_{\theta \in S_n} \text{tr}_g [X_{\theta_1}, \cdots, [X_{\theta_n}, \bullet] \cdots]
\]

where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$ are Bernoulli numbers.
Higher-dimensional simplices, $\sigma = \Delta^N$, $N \geq 2$: $\bar{l}_n, \bar{q}_n$ are given by a regularized homotopy transfer formula for transfer $g \otimes \Omega^\bullet(\Delta^N) \to g \otimes C^\bullet(\Delta^N)$
Higher-dimensional simplices, \( \sigma = \Delta^N, \ N \geq 2 \): \( \bar{l}_n, \bar{q}_n \) are given by a regularized homotopy transfer formula for transfer \( g \otimes \Omega^\bullet(\Delta^N) \rightarrow g \otimes C^\bullet(\Delta^N) \), with

- \( i \) = representation of cochains by Whitney elementary forms,
- \( p \) = integration over faces,
- \( s \) = Dupont’s chain homotopy operator.
Higher-dimensional simplices, \( \sigma = \Delta^N, \ N \geq 2 \): \( \bar{l}_n, \bar{q}_n \) are given by a regularized homotopy transfer formula for transfer \( g \otimes \Omega^\bullet(\Delta^N) \rightarrow g \otimes C^\bullet(\Delta^N) \), with

- \( i = \) representation of cochains by Whitney elementary forms,
- \( p = \) integration over faces,
- \( s = \) Dupont’s chain homotopy operator.

\[
\left\{ \bar{l}^\sigma_n, \bar{q}^\sigma_n \right\}(X_{\sigma_1}e_{\sigma_1}, \ldots, X_{\sigma_n}e_{\sigma_n}) = \sum_{\Gamma} C(\Gamma)^\sigma_{\sigma_1 \ldots \sigma_n} \text{Jacobi}_g(\Gamma; X_{\sigma_1}, \ldots, X_{\sigma_n})
\]

where \( \Gamma \) runs over binary rooted trees with \( n \) leaves for \( \bar{l}_n^\sigma \) and over trivalent 1-loop graphs with \( n \) leaves for \( \bar{q}_n^\sigma \); \( C(\Gamma)^\sigma_{\sigma_1 \ldots \sigma_n} \in \mathbb{R} \) are structure constants.
Higher-dimensional simplices, $\sigma = \Delta^N$, $N \geq 2$: $\bar{l}_n, \bar{q}_n$ are given by a regularized homotopy transfer formula for transfer $g \otimes \Omega^\bullet(\Delta^N) \rightarrow g \otimes C^\bullet(\Delta^N)$, with

- $i =$ representation of cochains by Whitney elementary forms,
- $p =$ integration over faces,
- $s =$ Dupont’s chain homotopy operator.

$$\bar{l}_n^\sigma \quad \bar{q}_n^\sigma \left\{ X_{\sigma_1} e_{\sigma_1}, \cdots, X_{\sigma_n} e_{\sigma_n} \right\} = \sum_{\Gamma} C(\Gamma)^\sigma_{\sigma_1 \cdots \sigma_n} \text{Jacobi}_g(\Gamma; X_{\sigma_1}, \cdots, X_{\sigma_n})$$

where $\Gamma$ runs over binary rooted trees with $n$ leaves for $\bar{l}_n^\sigma$ and over trivalent 1-loop graphs with $n$ leaves for $\bar{q}_n^\sigma$;
$C(\Gamma)^\sigma_{\sigma_1 \cdots \sigma_n} \in \mathbb{R}$ are structure constants.
There are explicit formulae for structure constants for small $n$. 
Summary: logic of the construction

building blocks $\bar{l}_n, \bar{q}_n$ on $\Delta^N$

\[
\begin{align*}
\downarrow & \quad \text{combinatorics of } T \\
\downarrow & \quad \text{algebraic structure on cochains} \\
\downarrow & \quad \text{homotopy transfer} \\
\downarrow & \quad \text{algebraic structure on cohomology}
\end{align*}
\]
Summary: logic of the construction

building blocks $\bar{l}_n, \bar{q}_n$ on $\Delta^N$

\[ \downarrow \text{combinatorics of } T \]

algebraic structure on cochains

\[ \downarrow \text{homotopy transfer} \]

algebraic structure on cohomology

- Operations $l_n$ on $g \otimes H^\bullet(T)$ are **Massey brackets** on cohomology and are a complete invariant of **rational homotopy type** in simply-connected case.

- Operations $q_n$ on $g \otimes H^\bullet(T)$ give a version of **Reidemeister torsion** of $T$.

- Construction above yields new local combinatorial formulae for Massey brackets (in other words: Massey brackets lift to a local algebraic structure on simplicial cochains).
Example: for a circle and a Klein bottle, $H^\bullet(S^1) \simeq H^\bullet(KB)$ as rings, but $\mathfrak{g} \otimes H^\bullet(S^1) \not\simeq \mathfrak{g} \otimes H^\bullet(KB)$ as unimodular $L_\infty$ algebras (distinguished by quantum operations).

\[
e^{\sum_n \frac{1}{n!} q_n (X \otimes \varepsilon, \cdots X \otimes \varepsilon)} =
\begin{cases}
\det_{\mathfrak{g}} \left( \frac{\sinh \frac{\text{ad} X}{2}}{\frac{\text{ad} X}{2}} \right) & \text{for } S^1 \\
\det_{\mathfrak{g}} \left( \frac{\text{ad} X}{2} \cdot \coth \frac{\text{ad} X}{2} \right)^{-1} & \text{for Klein bottle}
\end{cases}
\]

where $\varepsilon \in H^1$ – generator, $X \in \mathfrak{g}$ – variable.
Example: Massey bracket on the nilmanifold, combinatorial calculation

**Triangulation of the nilmanifold:**

- **one 0-simplex:** $A=B=C=D=A'=B'=C'=D'$
- **seven 1-simplices:** $AD=BC=A'D'=B'C'$, $AA'=BB'=CC'=DD'$, $AB=DC=D'B'$, $AC=A'B'=D'C'$, $AB'=DC'$, $AD'=BC'$, $AC'$
- **twelve 2-simplices:** $AA'B'=DD'C'$, $AB'B=DC'C$, $AA'D'=BB'C'$, $AD'=BC'C$, $ACD=AB'D'$, $ABC=D'B'C'$, $AB'D'$, $AC'D'$, $ACC'$, $ABC'$
- **six 3-simplices:** $AA'B'D'$, $AB'C'D'$, $ADC'D'$, $ABB'C'$, $ABCC'$, $ACDC'$

Massey bracket on $H_1$: $\mu_3(X \otimes [\alpha], Y \otimes [\beta], Z \otimes [\beta]) = \frac{1}{2} \mu_2(X \otimes [\alpha], Y \otimes [\beta]) + \sum_{\text{permutations of inputs}}$
Triangulation of the nilmanifold:

One 0-simplex: $A=B=C=D=A'=B'=C'=D'$

Seven 1-simplices: $AD=BC=A'D'=B'C'$,
$AA'=BB'=CC'=DD'$, $AB=DC=D'B'$,
$AC=A'B'=D'C'$, $AB'=DC'$, $AD'=BC'$, $AC'$

Twelve 2-simplices: $AA'B'=DD'C'$, $AB'B=DC'C$,
$AA'D'=BB'C'$, $AD'D=BC'C$, $ACD=AB'D'$,
$ABC=D'B'C'$, $AB'D'$, $AC'D'$, $ACC'$, $ABC'$

Six 3-simplices: $AA'B'D'$, $AB'C'D'$,
$ADC'D'$, $ABB'C'$, $ABCC'$, $ACDC'$

Massey bracket on $H^1$:

\[ l_3(X \otimes [\alpha], Y \otimes [\beta], Z \otimes [\beta]) = \]
\[ \frac{1}{2} X \otimes \alpha \xrightarrow{l_3^T} l_2^T - s^T + \frac{1}{6} Y \otimes \beta \xrightarrow{l_3^T} + \text{permutations of inputs} \]

\[ = ([X, Y], Z) + ([X, Z], Y) \otimes [\eta] \in g \otimes H^2(T) \]

Where $s^T = d^\vee / (dd^\vee + d^\vee d)$;
\[ \alpha = e_{AC} + e_{AD} + e_{AC'} + e_{AD'}, \beta = e_{AA'} + e_{AB'} + e_{AC'} + e_{AD'} \]
representatives of cohomology classes $[\alpha], [\beta]$ in simplicial cochains.
**Simplicial program for TFTs:** Given a TFT on a manifold $M$ with space of fields $F_M$ and action $S_M \in C^\infty(F_M)[[\hbar]]$, construct an exact discretization associating to a triangulation $T$ of $M$ a fin.dim. space $F_T$ and a local action $S_T \in C^\infty(F_T)[[\hbar]]$, such that partition function $Z_M$ and correlation functions can be obtained from $(F_T, S_T)$ by fin.dim. integrals. Also, if $T'$ is a subdivision of $T$, $S_T$ is an effective action for $S_{T'}$. 

M
TFT
M
T' TFT
M
T partition function
(invariant of M)
Example of a TFT for which the exact discretization exists: 

**BF theory:**

- **fields:** $F_M = \underbrace{g \otimes \Omega^1(M)}_{A} \oplus \underbrace{g^* \otimes \Omega^{\dim M-2}(M)}_{B}$,

- **action:** $S_M = \int_M \langle B \wedge dA + A \wedge A \rangle$,

- **equations of motion:** $dA + A \wedge A = 0$, $d_A B = 0$. 
### Algebra – TFT dictionary

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<th>$BF$ theory</th>
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<td>$BF_\infty$ theory, $F = V[1] \oplus V^*[-2]$, $S = \sum_n \frac{1}{n!} \langle B, l_n(A, \cdots, A) \rangle + \hbar \sum_n \frac{1}{n!} q_n(A, \cdots, A)$</td>
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<td>Quadratic relations on operations</td>
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<td>Homotopy transfer $V \to V'$</td>
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Batalin-Vilkovisky formalism


**Motivation:** resolution of the problem of degenerate critical loci in perturbation theory ("gauge-fixing").

**Definition**

A *BV algebra* \((A, \cdot, \{\}, \Delta)\) is a unital \(\mathbb{Z}\)-graded commutative algebra \((A^\bullet, \cdot, 1)\) endowed with:

- a degree 1 Poisson bracket \(\{\}, : A \otimes A \to A\) — a bi-derivation of \(\cdot\), satisfying Jacobi identity (i.e. \((A, \cdot, \{\}, \Delta)\) is a *Gerstenhaber algebra*),
- a degree 1 operator ("BV Laplacian") \(\Delta : A^\bullet \to A^{\bullet+1}\) satisfying

\[
\Delta^2 = 0, \quad \Delta(1) = 0, \quad \Delta(a \cdot b) = (\Delta a) \cdot b + (-1)^{|a|} a \cdot (\Delta b) + (-1)^{|a|} \{a, b\}
\]
Examples:

1. For $\mathcal{F}$ a $\mathbb{Z}$-graded manifold endowed with a degree $-1$ symplectic form $\omega$ and a “consistent” volume element $\mu$ (the data $(\mathcal{F}, \omega, \mu)$ is called an “$SP$-manifold”), the ring of functions $A = C^\infty(\mathcal{F})$ carries a BV algebra structure, with pointwise multiplication $\cdot$, and with

\[
\{f, g\} = \tilde{f}g, \quad \Delta f = \frac{1}{2} \text{div}_\mu \tilde{f}
\]

where $\tilde{f}$ is the Hamiltonian vector field for $f$ defined by $\iota_{\tilde{f}} \omega = df$. Consistency condition on $\mu$: $\Delta^2 = 0$.

2. Special case of the above when $(\mathcal{F}, \omega)$ is a degree $-1$ symplectic graded vector space and $\mu$ is the translation-invariant volume element.

3. Polyvector fields on a manifold $M$ carrying a volume element $\rho$, with opposite grading:

\[
A^\bullet = \mathcal{V}^{-\bullet}(M), \quad \cdot = \wedge, \quad \{,\} = [,,]_{NS}, \quad \Delta = \text{div}_\rho
\]

— this correspond to setting $\mathcal{F} = T^*[-1]M$ in (1).
Definition

Element $S \in A^0[[\hbar]]$ is said to satisfy Batalin-Vilkovisky quantum master equation (QME), if

$$\Delta e^{\frac{i}{\hbar}S} = 0$$

or equivalently in Maurer-Cartan form:

$$\frac{1}{2}\{S, S\} - i\hbar \Delta S = 0$$

Two solutions of QME, $S$ and $S'$ are said to be equivalent (related by a canonical transformation) if

$$e^{\frac{i}{\hbar}S'} = e^{\frac{i}{\hbar}S} + \Delta \left( e^{\frac{i}{\hbar}S} R \right)$$

for some generator $R \in A^{-1}[[\hbar]]$. For infinitesimal transformations:

$$S' = S + \{ S, R \} - i\hbar \Delta R$$
Fix an $SP$-manifold $(\mathcal{F}, \omega, \mu)$. Given a solution of QME $S \in C^\infty(\mathcal{F})[[\hbar]]$ and a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, one constructs the BV integral:

$$Z_{S, \mathcal{L}} = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S}$$

**BV-Stokes theorem (Batalin-Vilkovisky-Schwarz)**

1. If $\mathcal{L}, \mathcal{L}' \subset \mathcal{F}$ are two Lagrangian submanifolds that can be connected by a smooth family of Lagrangian submanifolds, then

$$Z_{S, \mathcal{L}} = Z_{S, \mathcal{L}'}$$

2. If $S$ and $S'$ are equivalent, then

$$Z_{S, \mathcal{L}} = Z_{S', \mathcal{L}}$$
Let \((F = F' \times F'', \omega = \omega' + \omega'', \mu = \mu' \times \mu'')\) be a product of two \(SP\)-manifolds and \(S\) a solution of QME on \(F\). Define the effective BV action \(S'\) on \(F'\) by the \textbf{fiberwise BV integral}

\[
e^{i\frac{\hbar}{\kappa}S'} = \int_{L'' \subset F''} e^{i\frac{\hbar}{\kappa}S}
\]

where \(L''\) is a Lagrangian submanifold of \(F''\).

**Theorem (P.M.)**

1. Effective BV action \(S'\) satisfies QME on \(F'\).
2. If \(L'', \tilde{L}'\) are two Lagrangian submanifolds of \(F''\) that can be connected by a smooth family of Lagrangian submanifolds, then corresponding effective actions are equivalent.
3. If \(S, \tilde{S}\) are two equivalent solutions of QME on \(F\), then the corresponding effective actions on \(F'\) are equivalent.
Thus the effective BV action construction defines the push-forward

\[(\text{solutions of QME on } \mathcal{F})/\text{equivalence}\]

\[
\downarrow \text{fiberwise BV integral}
\]

\[(\text{solutions of QME on } \mathcal{F}')/\text{equivalence}\]
One-dimensional simplicial Chern-Simons theory


Continuum theory on a circle. Fix \((g, \langle, \rangle)\) be a *quadratic* even-dimensional Lie algebra.

- Space of fields: \(\mathcal{F} = \Pi g \otimes \Omega^0(S^1) \oplus g \otimes \Omega^1(S^1)\) — a \(\mathbb{Z}_2\)-graded manifold with an odd symplectic structure coming from Poincaré duality on \(S^1\): \(\omega = \int_{S^1} \langle \delta \psi \wedge \delta A \rangle\)

- Action: \(S(\psi, A) = \int_{S^1} \langle \psi \wedge d\psi + [A, \psi] \rangle\)

Effective BV action on cochains of triangulated circle.
Denote \(T_N\) the triangulation of \(S^1\) with \(N\) vertices. Discrete space of fields:

\[\mathcal{F}_{T_N} = \Pi g \otimes C^0(T_N) \oplus g \otimes C^1(T_N)\]

with coordinates \(\{\psi_k \in \Pi g, A_k \in g^1\}_{k=1}^N\) and odd symplectic form

\[\omega_{T_N} = \sum_{k=1}^N \langle \delta \left( \frac{\psi_k + \psi_{k+1}}{2} \right), \delta A_k \rangle\]
Explicit simplicial Chern-Simons action on cochains of triangulated circle:

\[
S_{TN} = \left(-\frac{1}{2}\sum_{k=1}^{N} \left( \psi_k, \psi_{k+1} \right) + \frac{1}{3} \left( \psi_k, \text{ad}_{A_k} \psi_k \right) + \frac{1}{3} \left( \psi_{k+1}, \text{ad}_{A_k} \psi_{k+1} \right) + \frac{1}{3} \left( \psi_k, \text{ad}_{A_k} \psi_{k+1} \right) + \frac{1}{2} \right) \\
+ \frac{1}{2} \sum_{k=1}^{N} (\psi_{k+1} - \psi_k) \left( \frac{1 - R(\text{ad}_{A_k})}{2} \left( \frac{1}{1 + \mu_k(A')} - \frac{1}{1 + R(\text{ad}_{A_k})} \right) \right) \cdot \frac{1 - R(\text{ad}_{A_k})}{2} \\
+ (\text{ad}_{A_k})^{-1} + \frac{1}{12} \text{ad}_{A_k} - \frac{1}{2} \coth \frac{\text{ad}_{A_k}}{2} \circ (\psi_{k+1} - \psi_k) + \\
+ \frac{1}{2} \sum_{k'=1}^{k'-N-1} \sum_{k=k'+1}^{N-k'-1} (-1)^{k-k'} (\psi_{k+1} - \psi_k) \cdot \frac{1 - R(\text{ad}_{A_k})}{2} \cdot R(\text{ad}_{A_{k-1}}) \cdots R(\text{ad}_{A_{k'}}) \cdot \\
\cdot \frac{1}{1 + \mu_{k'}(A')} \cdot \frac{1 - R(\text{ad}_{A_{k'}})}{2} \circ (\psi_{k'+1} - \psi_{k'}) + \\
+ \hbar \frac{1}{2} \text{tr}_g \log \left( (1 + \mu \cdot (A')) \prod_{k=1}^{n} \left( \frac{1}{1 + R(\text{ad}_{A_k})} \cdot \frac{\sinh \frac{\text{ad}_{A_k}}{2}}{\text{ad}_{A_k}} \right) \right) \right)
\]

where

\[
R(A) = -\frac{A^{-1} + \frac{1}{2} - \frac{1}{2} \coth \frac{A}{2}}{A^{-1} - \frac{1}{2} - \frac{1}{2} \coth \frac{A}{2}}, \quad \mu_k(A') = R(\text{ad}_{A_{k-1}}) R(\text{ad}_{A_{k-2}}) \cdots R(\text{ad}_{A_{k+1}}) R(\text{ad}_{A_k})
\]
Questions:

- Why such a long formula?
- It is not simplicially local (there are monomials involving distant simplices). How to disassemble the result into contributions of individual simplices?
- How to check quantum master equation for $S_{TN}$ explicitly?
- Simplicial aggregations should be given by finite-dimensional BV integrals; how to check that?
1D simplicial Chern-Simons as Atiyah’s TFT

Set

$$\zeta(\tilde{\psi}, A) = (i\hbar)^{-\dim g/2} \int_{\Pi g} D\lambda \exp \left( -\frac{1}{2\hbar} \langle \hat{\psi}, [A, \hat{\psi}] \rangle + \langle \lambda, \hat{\psi} - \tilde{\psi} \rangle \right) \in Cl(g)$$

where \(\{\hat{\psi}^a\}\) are generators of the Clifford algebra \(Cl(g)\),

\[\hat{\psi}^a \hat{\psi}^b + \hat{\psi}^b \hat{\psi}^a = \hbar \delta^{ab}\]

Element \(\zeta\) can be used as a building block (partition function for an interval with standard triangulation) for 1D Chern-Simons as Atiyah’s TFT on triangulated 1-cobordisms \(\Theta\), with

- Partition functions
  
  $$Z_\Theta \in C^\infty(\Pi g \otimes C^1(\Theta) \oplus g \otimes C^1(\Theta)) \otimes Cl(g) \otimes \# \{\text{intervals}\},$$

- For a disjoint union, \(Z_{\Theta_1 \sqcup \Theta_2} = Z_{\Theta_1} \otimes Z_{\Theta_2}\),

- For a concatenation of two triangulated intervals, \(Z_{\Theta_1 \cup \Theta_2} = Z_{\Theta_1} * Z_{\Theta_2} - \text{Clifford product}\),

- For the closure of a triangulated interval \(\Theta\) into a triangulated circle \(\Theta'\), \(Z_{\Theta'} = \text{Str}_{Cl(g)} Z_{\Theta} - \text{Clifford supertrace}\).
Theorem (A. Alekseev, P.M.)

1. For a triangulated circle,
\[ Z_{TN} = \text{Str}_{Cl(g)} \left( \zeta(\tilde{\psi}_N, A_N) \ast \cdots \ast \zeta(\tilde{\psi}_1, A_1) \right) = e^{\frac{i}{\hbar}S_{TN}} \]

2. For a triangulated interval, the partition function satisfies the \textit{modified} quantum master equation
\[ \hbar \Delta_\Theta Z_\Theta + \frac{1}{\hbar} \left[ \frac{1}{6} \langle \hat{\psi}, [\hat{\psi}, \hat{\psi}] \rangle, Z_\Theta \right]_{Cl(g)} = 0 \]

where \( \Delta_\Theta = \sum_k \frac{\partial}{\partial \psi_k} \frac{\partial}{\partial A_k} \).

3. Simplicial action on triangulated circle \( S_{TN} \) satisfies the usual BV quantum master equation, \( \Delta_{TN} e^{\frac{i}{\hbar}S_{TN}} = 0 \).

The space of states for a point. Fix a complex polarization \( g \otimes \mathbb{C} = \mathfrak{h} \oplus \mathfrak{\bar{h}} \). Then one has an isomorphism \( \rho : Cl(g) \rightarrow C^\infty(\Pi \mathfrak{h}) \otimes C^\infty(\Pi \mathfrak{\bar{h}}) \). Thus we set
\[ \mathcal{H}_{pt^+} = C^\infty(\Pi \mathfrak{h}), \quad \mathcal{H}_{pt^-} = C^\infty(\Pi \mathfrak{\bar{h}}) \simeq (\mathcal{H}_{pt^+})^* \]
The building block $\zeta$ can be written as a path integral with boundary conditions:

$$
\rho(\zeta)(\eta_{\text{out}}, \bar{\eta}_{\text{in}}; \bar{\psi}, A) = \int_{\pi_{\text{out}} = \eta_{\text{out}}, \bar{\pi}_{\text{in}} = \bar{\eta}_{\text{in}}, \int_{0}^{1} d\psi = \bar{\psi}} \mathcal{D}\psi \, e^{\frac{i}{\hbar} \int_{0}^{1} \langle \psi^\dagger \, d\psi + [A d\psi, \psi] \rangle}
$$

where $\pi : \mathbf{g}_{\mathbb{C}} \to \mathfrak{h}$, $\bar{\pi} : \mathbf{g}_{\mathbb{C}} \to \bar{\mathfrak{h}}$ are the projections to the two terms in $\mathbf{g}_{\mathbb{C}} \simeq \mathfrak{h} \oplus \bar{\mathfrak{h}}$. 
Classical BV structure for gauge theory on a closed manifold:
A graded manifold $\mathcal{F}$ (space of fields) endowed with
- a cohomological vector field $Q$ of degree 1, $Q^2 = 0$,
- a degree $-1$ symplectic form $\omega$,
- a degree 0 Hamiltonian function $S$ generating the cohomological vector field: $\delta S = \iota_Q \omega$

Extension to manifolds with boundary ("BV-BFV formalism").
To a manifold $\Sigma$ with boundary $\partial \Sigma$ a gauge theory associates:
- **Boundary BFV data:** a graded manifold $\mathcal{F}_{\partial}$ endowed with
  - a degree 1 cohomological vector field $Q_{\partial}$,
  - a degree 0 exact symplectic form $\omega_{\partial} = \delta \alpha_{\partial}$,
  - a degree 1 Hamiltonian $S_{\partial}$ generating $Q_{\partial}$, i.e. $Q_{\partial} = \{S_{\partial}, \bullet\}_{\omega_{\partial}}$.
- **Bulk BV data:** a graded manifold $\mathcal{F}$ endowed with
  - a degree 1 cohomological vector field $Q$,
  - a projection $\pi : \mathcal{F} \to \mathcal{F}_{\partial}$ which is a $Q$-morphism, i.e. $d\pi(Q) = Q_{\partial}$,
  - a degree $-1$ symplectic form $\omega$,
  - a degree 0 function $S$ satisfying $\delta S = \iota_Q \omega + \pi^* \alpha_{\partial}$.

Euler-Lagrange spaces.
One can define coisotropic submanifolds \( \mathcal{E}L \subset \mathcal{F} \), \( \mathcal{E}L_\partial \subset \mathcal{F}_\partial \) as zero loci of \( Q \) and \( Q_\partial \) respectively. For “nice” theories, the “evolution relation” \( \mathcal{L} = \pi(\mathcal{E}L) \subset \mathcal{E}L_\partial \subset \mathcal{F}_\partial \) is Lagrangian.

Reduction: EL moduli spaces.
One can quotient Euler-Lagrange spaces by the distribution induced from the cohomological vector field to produce \( EL \) moduli spaces \( \mathcal{M} = \mathcal{E}L/Q \), \( \mathcal{M}_\partial = \mathcal{E}L/Q_\partial \). They carry the following structure induced from BV-BFV structure on fields:

- map \( \pi_* : \mathcal{M} \to \mathcal{M}_\partial \),
- \( \mathcal{M}_\partial \) is degree 0 symplectic, \( \mathcal{M} \) is degree 1 Poisson,
- image of \( \pi_* \) is Lagrangian, fibers of \( \pi_* \) comprise the symplectic foliation of \( \mathcal{M} \),
- a line bundle \( L \) over \( \mathcal{M}_\partial \) with connection \( \nabla \) of curvature being the symplectic form on \( \mathcal{M}_\partial \),
- a horizontal section of the pull-back bundle \( (\pi_*)^* L \).
A simple example: abelian Chern-Simons theory on a 3-manifold $\Sigma$ with boundary.

$$\mathcal{F} = \Omega^\bullet(\Sigma), \quad S = \frac{1}{2} \int_{\Sigma} A \wedge dA, \quad \omega = \frac{1}{2} \int_{\Sigma} \delta A \wedge \delta A,$$

$$\mathcal{F}_\partial = \Omega^\bullet(\partial \Sigma), \quad S_\partial = \frac{1}{2} \int_{\partial \Sigma} A_\partial \wedge dA_\partial, \quad \alpha_\partial = \frac{1}{2} \int_{\partial \Sigma} A_\partial \wedge \delta A_\partial$$

Euler-Lagrange spaces: $\mathcal{EL} = \Omega^\bullet_{\text{closed}}(\Sigma), \mathcal{EL}_\partial = \Omega^\bullet_{\text{closed}}(\partial \Sigma)$.
EL moduli spaces: $\mathcal{M} = H^\bullet(\Sigma), \mathcal{M}_\partial = H^\bullet(\partial \Sigma)$.

**Non-abelian Chern-Simons theory.** EL moduli spaces are (derived versions of) the moduli spaces of flat $G$-bundles over $\Sigma$ and $\partial \Sigma$.

**Remarks:**

- One can introduce the third EL moduli space $\mathcal{M}_{\text{rel}}$, so that the triple $(\mathcal{M}_{\text{rel}}, \mathcal{M}, \mathcal{M}_\partial)$ supports long exact sequence for tangent spaces, Lefschetz duality, Meyer-Vietoris type gluing.

- EL moduli spaces come with a cohomological description, $\mathcal{M} = \text{Spec } H_Q(C^\infty(\mathcal{F}))$ which is particularly useful for quantization. (E.g. we get a simple cohomological description of Verlinde space, arising as the geometric quantization of the moduli space of local systems).
Idea of quantization.
Take a foliation of $\mathcal{F}_\partial$ by Lagrangian submanifolds. Each leaf of the foliation is a valid boundary condition for bulk fields in the path integral. Space of states is constructed as

$$\mathcal{H}_{\partial \Sigma} = \text{Fun}\{\text{space of leaves of the foliation}\}$$

with a differential $\hat{S}_\partial$. Partition function, constructed by the path integral, is a function of the leaf and of the bulk zero-modes (i.e. function on fiber of $\pi_* : \mathcal{M} \to \mathcal{M}_\partial$), and is expected to satisfy a version of quantum master equation:

$$(\Delta_{\text{bulk z.m.}} + \hat{S}_\partial)Z_\Sigma = 0$$
Developments

- Axelrod-Singer’s perturbative treatment of Chern-Simons on closed manifolds extended to non-acyclic background flat connections. Algebraic model of Chern-Simons based on dg Frobenius algebras studied. 

- Global perturbation theory for Poisson sigma model studied from the standpoint of formal geometry of the target. Genus 1 partition function with Kähler target is shown yield Euler characteristic of the target. 

- A class of generalized Wilson loop observables constructed via BV push-forward of the transgression of a Hamiltonian $Q$-bundle over the target to the mapping space. 

- Cohomology of $\hat{S}_\phi$ on the canonical quantization of boundary BFV phase space of Chern-Simons with Wilson lines yields the space of conformal blocks of Wess-Zumino-Witten model. 
Program

- Construct perturbative quantization of TFTs in the BV-BFV formalism as a (far-reaching) extension of Axelrod-Singer’s construction. Possible application: link between Reshetikhin-Turaev invariant and Chern-Simons theory.

- Study applications to invariants of manifolds and knots consistent with surgery. (In particular, study the extension of gluing formulae for cohomology and Ray-Singer torsion to higher perturbative invariants, e.g. Axelrod-Singer and Bott-Cattaneo invariants of 3-manifolds.)

- Further study of EL moduli spaces (and their geometric quantization) from the point of view of derived symplectic geometry.

- Extend the construction to allow manifolds with corners; compare the results with Baez-Dolan-Lurie axioms for extended TFTs.