BV pushforwards and exact discretizations in topological field theory

Pavel Mnev

Max Planck Institute for Mathematics, Bonn

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Manifold $\longrightarrow$ Invariants of the manifold

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Lower left way: mathematical physics/topological field theory (Schwarz, Witten, Kontsevich,...)

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Upper right way: algebraic topology (Poincaré, de Rham,...)
Lower left way: mathematical physics/topological field theory (Schwarz, Witten, Kontsevich,...)
What happens when we replace M with its combinatorial description? (E.g. a triangulation)

Pushforward in probability theory:

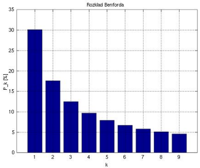
$y = F(x)$

x has probability distribution μ

implies y has probability distribution $F_* \mu$.

Examples:

1. Throw two dice. What is the distribution for the sum?
2. Benford's law.



Pushforward in geometry: fiber integral.

Plan.

- From discrete forms on the interval to Batalin-Vilkovisky formalism
- Effective action (BV pushforward)
- Application to topological field theory

Appetizer/warm-up problem:
discretize the algebra of differential forms on the interval $I = [0,1]$.
De Rham algebra $\Omega^\bullet(I) \ni f(t) + g(t) \cdot dt$ with operations $d, \wedge$ satisfying

- $d^2 = 0$
- Leibniz rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$
- Associativity $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

Also: super-commutativity $\alpha \wedge \beta = \pm \beta \wedge \alpha$.

The problem: construct the algebra structure on "discrete forms" (cellular cochains) $C^\bullet(I) = \mathrm{Span}(e_0, e_1, e_{01}) \ni a \cdot e_0 + b \cdot e_1 + c \cdot e_{01}$ with same properties.
Represent generators by forms

$$i: \quad e_0 \mapsto 1-t, \quad e_1 \mapsto t, \quad e_{01} \mapsto dt$$

And define a projection

$$p: \quad f(t) + g(t)\cdot dt \quad \mapsto \quad f(0)\cdot e_0 + f(1)\cdot e_1 + \left(\int_0^1 g(\tau)d\tau\right)\cdot e_{01}$$

Construct d and $\wedge$ on $C^\bullet$:

- $\boxed{d = p \circ d \circ i}$, i.e.

$$d(e_0) = -e_{01}, \quad d(e_1) = e_{01}, \quad d(e_{01}) = 0$$

- $\boxed{\alpha \wedge \beta = p(i(\alpha) \wedge i(\beta))}$, i.e.

$$e_0 \wedge e_0 = e_0, \; e_1 \wedge e_1 = e_1, \; e_0 \wedge e_{01} = \frac{1}{2} e_{01}, \; e_1 \wedge e_{01} = \frac{1}{2} e_{01}, \; e_{01} \wedge e_{01} = 0$$

$d, \wedge$ satisfy $d^2 = 0$, Leibniz, but associativity fails:

$$e_0 \wedge (e_0 \wedge e_{01}) \neq (e_0 \wedge e_0) \wedge e_{01}$$

However, one can introduce a trilinear operation m_3 such that

$$\alpha \wedge (\beta \wedge \gamma) - (\alpha \wedge \beta) \wedge \gamma = \\ = d\, m_3(\alpha, \beta, \gamma) \pm m_3(d\alpha, \beta, \gamma) \pm m_3(\alpha, d\beta, \gamma) \pm m_3(\alpha, \beta, d\gamma)$$

– "associativity up to homotopy".
m_3 itself satisfies

$$[\wedge, m_3] = -[d, m_4]$$

for some 4-linear operation m_4 etc.
– a sequence of operations $(m_1 = d, m_2 = \wedge, m_3, m_4, \ldots)$ satisfying a sequence of homotopy associativity relations – an A_∞ algebra structure on $C^\bullet(I)$.

Aside: A_∞ algebras

Definition (Stasheff)

An A_∞ algebra is:

1. a $\mathbb{Z}$-graded vector space $V^\bullet$,
2. a set of multilinear operations $m_n : V^{\otimes n} \to V$, $n \geq 1$,

satisfying the set of quadratic relations

$$\sum_{q+r+s=n} m_{q+s+1}(\underbrace{\bullet, \cdots, \bullet}_{q}, m_r(\underbrace{\bullet, \cdots, \bullet}_{r}), \underbrace{\bullet, \cdots, \bullet}_{s}) = 0$$

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Remark:

- Case $m_{\neq 2} = 0$ – associative algebra.
- Case $m_{\neq 1,2} = 0$ – differential graded associative algebra (DGA).

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- Case $m_{\neq 2} = 0$ – associative algebra.
- Case $m_{\neq 1,2} = 0$ – differential graded associative algebra (DGA). Examples:
  1. Singular cochains of a topological space $C^\bullet_{\text{sing}}(X)$ – non-commutative DGA.
  2. De Rham algebra of a manifold $\Omega^\bullet(M)$ – super-commutative DGA.

Motivating example: Cohomology of a top. space $H^\bullet(X)$ carries a natural A_∞ algebra structure, with

- $m_1 = 0$,
- m_2 the cup product,
- $m_3, m_4, \cdots$ the (higher) Massey products on $H^\bullet(X)$.

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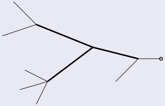
Quillen, Sullivan: this A_∞ structure encodes the data of rational homotopy type of X, i.e. rational homotopy groups $\mathbb{Q} \otimes \pi_k(X)$ can be recovered from $\{m_n\}$.

Homotopy transfer theorem for A_∞ algebras (Kadeishvili, Kontsevich-Soibleman)

If $(V^\bullet, \{m_n\})$ is an A_∞ algebra and $V' \hookrightarrow V$ a deformation retract of (V, m_1), then V' carries an A_∞ structure with

$m'_n = \sum_T$  $: (V')^{\otimes n} \to V'$

where T runs over rooted trees with n leaves.

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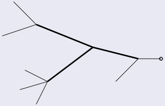
$m'_n = \sum_T$  $: (V')^{\otimes n} \to V'$

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Decorations:

leaf	$i : V' \hookrightarrow V$		root	$p : V \twoheadrightarrow V'$
edge	$-s : V^\bullet \to V^{\bullet-1}$		$(k+1)$-valent vertex	m_k

where s is a chain homotopy, $m_1\, s + s\, m_1 = \mathrm{id} - i\, p$.

Example: $V = \Omega^\bullet(M), d, \wedge$ the de Rham algebra of a Riemannian manifold (M, g),
$V' = H^\bullet(M)$ de Rham cohomology realized by harmonic forms. Induced (transferred) A_∞ algebra gives Massey products.

Back to the A_∞ algebra on cochains of the interval.

Explicit answer for algebra operations:

$$m_{n+1}(\underbrace{e_{01},\ldots,e_{01}}_{k},e_1,\underbrace{e_{01},\ldots,e_{01}}_{n-k}) = \pm \begin{pmatrix} n \\ k \end{pmatrix} \cdot B_n \cdot e_{01}$$

(and similarly for $e_1 \leftrightarrow e_0$), where
$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots$ are Bernoulli numbers, i.e. coefficients of $\frac{x}{e^x - 1} = \sum_{n\geq 0} \frac{B_n}{n!} x^n$.

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This is a special case of homotopy transfer of algebraic structures (Kontsevich-Soibelman,...), $(\Omega^\bullet(I), d, \wedge) \to (C^\bullet(I), m_1, m_2, m_3, \cdots)$.
Another point of view (Losev-P.M.): this result comes from a calculation of a particular path integral, and Bernoulli numbers arise as values of certain Feynman diagrams!

Allow coefficients of cochains to be matrices $N \times N$, or elements of a more general Lie algebra $\mathfrak{g}$. Then we get an L_∞ algebra structure on $C^\bullet(I, \mathfrak{g})$, with skew-symmetric multilinear operations $(l_1 = d, l_2 = [,], l_3, l_4, \ldots)$ satisfying a sequence of homotopy Jacobi identities.

$$\Omega^\bullet(I) \otimes \mathfrak{g}, \quad d, \quad [\ \overset{\wedge}{,}\ ] \quad \longrightarrow \quad C^\bullet(I) \otimes \mathfrak{g}, \quad \{l_n\}_{n \geq 1}$$

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Definition (Lada-Stasheff)

An L_∞ algebra is:

1. a $\mathbb{Z}$-graded vector space $V^\bullet$,
2. a set of skew-symmetric multilinear operations $l_n : \wedge^n V \to V$, $n \geq 1$,

satisfying the set of quadratic relations

$$\sum_{r+s=n} \frac{1}{r!s!} l_{r+1}(\underbrace{\bullet,\cdots,\bullet}_{r}, l_s(\underbrace{\bullet,\cdots,\bullet}_{s})) = 0$$

with skew-symmetrization over all inputs implied.

An L_∞ algebra structure on a graded vector space $V^\bullet$ can be packaged into a **generating function** (the **master action**)

$$S(A,B) = \sum_{n\geq 1} \frac{1}{n!} \langle B, l_n(\underbrace{A,\ldots,A}_{n})\rangle$$

where $A, B \in V[1] \oplus V^*[-2]$ are the variables – **fields**.
Quadratic relations on operations l_n are packaged into the **Batalin-Vilkoviski (classical) master equation**

$$\boxed{\{S,S\} = 0}$$

where $\{f,g\} = \sum_i \frac{\partial f}{\partial A^i}\frac{\partial g}{\partial B_i} - \frac{\partial f}{\partial B_i}\frac{\partial g}{\partial A^i}$ is the **odd Poisson bracket**.

Several classes of algebraic/geometric structures can be packaged into solutions of the master equation (allowing for different parities of $\{,\}$, S):

- Lie and L_∞ algebras
- quadratic Lie and cyclic L_∞ algebras
- representation of a Lie algebra, "representation up to homotopy"
- Lie algebroids
- Courant algebroids
- Poisson manifolds
- differential graded manifolds
- coisotropic submanifold of a symplectic manifold

Classical master equation (CME) $\{S, S\} = 0$ is the leading term of the Quantum master equation (QME)

$$\{S_\hbar, S_\hbar\} - 2i\hbar\Delta S_\hbar \quad \Leftrightarrow \quad \Delta e^{\frac{i}{\hbar}S_\hbar} = 0$$

on $S_\hbar = S + S^{(1)}\hbar + S^{(2)}\hbar^2 + \cdots \in C^\infty(Fields)[[\hbar]]$, where

$$\Delta = \sum_i \frac{\partial}{\partial A^i}\frac{\partial}{\partial B_i}$$

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Example. (P.M.) Solution of CME corresponding to the discrete forms on the interval extends (uniquely!) to a solution of QME:

$$S_\hbar=\langle B_0,\frac{1}{2}[A_0,A_0]\rangle+\langle B_1,\frac{1}{2}[A_1,A_1]\rangle+$$
$$+\langle B_{01},\left[A_{01},\frac{A_0+A_1}{2}\right]+\mathsf{F}([A_{01},\bullet])\circ(A_1-A_0)\rangle\underbrace{-i\hbar\,\log\det\nolimits_{\mathfrak{g}}\mathsf{G}([A_{01},\bullet])}_{\hbar-\text{correction}}$$

where

$$\mathsf{F}(x)=\frac{x}{2}\coth\frac{x}{2},\quad \mathsf{G}(x)=\frac{2}{x}\sinh\frac{x}{2}$$

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where

$$\mathsf{F}(x) = \frac{x}{2}\coth\frac{x}{2}, \quad \mathsf{G}(x) = \frac{2}{x}\sinh\frac{x}{2}$$

$S_\hbar$ generates the unimodular (or quantum) L_∞ structure on $C^\bullet(I,\mathfrak{g})$.

Definition (Granåker, P.M.)

A unimodular L_∞ algebra is:

- an L_∞ algebra $V, \{l_n\}_{n\geq 1}$, endowed additionally with
- "quantum operations" $q_n : \wedge^n V \to \mathbb{R}$, $n \geq 1$,

satisfying, in addition to L_∞ relations,

$$\frac{1}{n!}\mathrm{Str}\, l_{n+1}(\bullet, \cdots, \bullet, -)+ \\ + \sum_{r+s=n} \frac{1}{r!s!} q_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$$

(with inputs skew-symmetrized).

Summary of BV structure:

- $\mathbb{Z}$-graded vector space of fields $\mathcal{F}$,
- symplectic structure (BV 2-form) ω on $\mathcal{F}$ of degree $\mathrm{gh}\,\omega = -1$ – induces $\{,\}$ and Δ on $C^\infty(\mathcal{F})$,
- action $S \in C^\infty(\mathcal{F})[[\hbar]]$ – a solution of QME

$$\boxed{\Delta e^{\frac{i}{\hbar}S} = 0}$$

Construction (Costello, Losev, P.M.): pushforward of solutions of QME — BV pushforward/effective BV action/fiber BV integral.
Let

- $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ – splitting compatible with $\omega = \omega' \oplus \omega''$,
- $\mathcal{L} \subset \mathcal{F}''$ a Lagrangian subspace

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Define $S' \in C^\infty(\mathcal{F}')[[\hbar]]$ by

$$\boxed{e^{\frac{i}{\hbar}S'(x';\hbar)} = \int_{\mathcal{L}\ni x''} e^{\frac{i}{\hbar}S(x'+x'';\hbar)}}$$

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Remark: to make sense of this, we need reference half-densities μ, μ', μ'' on $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ with $\mu = \mu' \cdot \mu''$. Correct formula:

$$e^{\frac{i}{\hbar}S'(x';\hbar)}\mu' = \int_{\mathcal{L}\ni x''} e^{\frac{i}{\hbar}S(x'+x'';\hbar)} \underbrace{\mu|_{\mathcal{L}}}_{\mu'\cdot\mu''|_{\mathcal{L}}}$$

Properties:

- If S satisfies QME on $\mathcal{F}$ then S' satisfies QME on on $\mathcal{F}'$,
- if $\widetilde{S}$ is equivalent (homotopic) to S, i.e. $e^{\frac{i}{\hbar}\widetilde{S}} - e^{\frac{i}{\hbar}S} = \Delta(\cdots)$, then the corresponding BV pushforwards $\widetilde{S}'$ and S' are equivalent.
- If $\widetilde{\mathcal{L}}$ is a Lagrangian homotopic to $\mathcal{L}$ in $\mathcal{F}''$, then the corresponding BV pushforward $\widetilde{S}'$ is equivalent to S' (obtained with $\mathcal{L}$).

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Notation: $e^{\frac{i}{\hbar}S'} = P_*^{(\mathcal{L})}\left(e^{\frac{i}{\hbar}S}\right)$.

(Here $P : \mathcal{F} \twoheadrightarrow \mathcal{F}'$.)

Example: Reidemeister torsion as BV pushforward.
Reference: P. Mnev, "Lecture notes on torsions," arXiv:1406.3705 [math.AT],
A. S. Cattaneo, P. Mnev, N. Reshetikhin, "Cellular BV-BFV-BF theory," in preparation.
Input: X - cellular complex, $\rho : \pi_1(X) \to O(m)$ local system.

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Define a BV system $\mathcal{F} = V[1] \oplus V^*[-2] \ni (A, B)$, $\boxed{S = \langle B, d_\rho A \rangle}$.

Induce onto cohomology $\mathcal{F}' = H^\bullet[1] \oplus (H^\bullet)^*[-2]$.

Example: Reidemeister torsion as BV pushforward.

Result of BV pushforward:

$$P_*(e^{\frac{i}{\hbar}S}) = \zeta \cdot \tau(X,\rho) \quad \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}' \cong \mathrm{Det}H^\bullet/\{\pm 1\}.$$

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- $\tau(X,\rho) \in \mathrm{Det}H^\bullet/\{\pm 1\}$ — the Reidemeister torsion (an invariant of simple homotopy type of X, in particular invariant under subdivisions of X).
- $\zeta = (2\pi\hbar)^{\frac{\dim \mathcal{L}^{\mathrm{even}}}{2}} \cdot (\frac{i}{\hbar})^{\frac{\dim \mathcal{L}^{\mathrm{odd}}}{2}} = \frac{\xi^{H^\bullet}}{\xi^{C^\bullet}} \quad \in \mathbb{C}$. Here $\xi^{H^\bullet} = (2\pi\hbar)^{\sum_k (-\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot \dim H^k} \cdot (e^{-\frac{\pi i}{2}}\hbar)^{\sum_k (\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot \dim H^k}$ – a topological invariant, $\xi^{C^\bullet}$ – "extensive" (multiplicative in numbers of cells).

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- $\zeta = (2\pi\hbar)^{\frac{\dim\mathcal{L}^{\mathrm{even}}}{2}}\cdot(\frac{i}{\hbar})^{\frac{\dim\mathcal{L}^{\mathrm{odd}}}{2}} = \frac{\xi^{H^\bullet}}{\xi^{C^\bullet}} \quad \in\mathbb{C}$. Here $\xi^{H^\bullet} = (2\pi\hbar)^{\sum_k(-\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot\dim H^k}\cdot(e^{-\frac{\pi i}{2}}\hbar)^{\sum_k(\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot\dim H^k}$ – a topological invariant, $\xi^{C^\bullet}$ – "extensive" (multiplicative in numbers of cells).

Thus
$$P_*(e^{\frac{i}{\hbar}S}\cdot\underbrace{\xi^{C^\bullet}}_{\text{correction to }\mu}) = \xi^{H^\bullet}\cdot\tau(X,\rho)$$

— a topological invariant, contains a mod 16 phase.

Aside: perturbed Gaussian integrals. (After Feynman, Dyson).
Let W vector space with fixed basis, $B(x,y) = B_{ij}x^i x^j$ non-degenerate bilinear form on W, $p(x) = \sum_k \frac{(p_k)_{i_1\cdots i_k}}{k!} x^{i_1}\cdots x^{i_k}$ a polynomial.

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One has the following asymptotic equivalence as $\hbar \to 0$:

$$\int_W dx \cdot e^{\frac{i}{\hbar}\left(\frac{1}{2}B(x,x)+p(x)\right)} \underset{\hbar\to 0}{\sim}$$

$$\sim (2\pi\hbar)^{\frac{1}{2}\dim W} e^{\frac{\pi i}{4}\mathrm{sgn}(B)} \cdot (\det B)^{-\frac{1}{2}} \cdot \exp \frac{i}{\hbar}\sum_\Gamma \frac{(-i\hbar)^{\mathrm{loops}(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \cdot \Phi_\Gamma$$

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This gives a way to define (special) infinite-dimensional integrals in terms of "Feynman diagrams" Γ.

Homotopy transfer as BV pushforward (Losev, P.M.)

algebra	associated BV package
unimodular DGLA $V^\bullet, d, [,]$	

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Perturbative (Feynman diagram) computation of the BV pushforward yields the Kontsevich-Soibelman sum-over-trees formula for classical L_∞ operations l'_n, and a formula involving 1-loop graphs for induced "quantum operations" q'_n.

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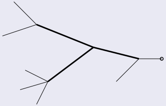
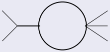
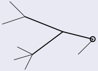
Perturbative (Feynman diagram) computation of the BV pushforward yields the Kontsevich-Soibelman sum-over-trees formula for classical L_∞ operations l'_n, and a formula involving 1-loop graphs for induced "quantum operations" q'_n.
Instead of starting with a uDGLA, one can start with a uL_∞ algebra.

Homotopy transfer theorem (P.M.)

If $(V, \{l_n\}, \{q_n\})$ is a unimodular L_∞ algebra and $V' \hookrightarrow V$ is a deformation retract of (V, l_1), then

1. V' carries a unimodular L_∞ structure given by

$l'_n = \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to V'$

$q'_n = \sum_{\Gamma_1} \frac{1}{|\mathrm{Aut}(\Gamma_1)|}$  $+ \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to \mathbb{R}$

where Γ_0 runs over rooted trees with n leaves and Γ_1 runs over 1-loop graphs with n leaves.

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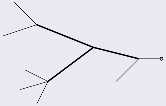
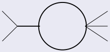
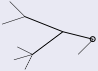
$l'_n = \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to V'$

$q'_n = \sum_{\Gamma_1} \frac{1}{|\mathrm{Aut}(\Gamma_1)|}$  $+ \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to \mathbb{R}$

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leaf	$i : V' \hookrightarrow V$	root	$p : V \twoheadrightarrow V'$
edge	$-s : V^\bullet \to V^{\bullet-1}$	$(m+1)$-valent vertex	l_m
cycle	super-trace over V	m-valent $\circ$-vertex	q_m

where s is a chain homotopy, $l_1\, s + s\, l_1 = \mathrm{id} - i\, p$.

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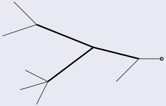
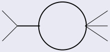
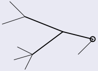
   $l'_n = \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to V'$

   $q'_n = \sum_{\Gamma_1} \frac{1}{|\mathrm{Aut}(\Gamma_1)|}$  $+ \sum_{\Gamma_0} \frac{1}{|\mathrm{Aut}(\Gamma_0)|}$  $: \wedge^n V' \to \mathbb{R}$

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2. Algebra $(V', \{l'_n\}, \{q'_n\})$ changes by isomorphisms under changes of induction data (i, p, s).

Topological field theory (Lagrangian formalism).
On a manifold M, classically one has

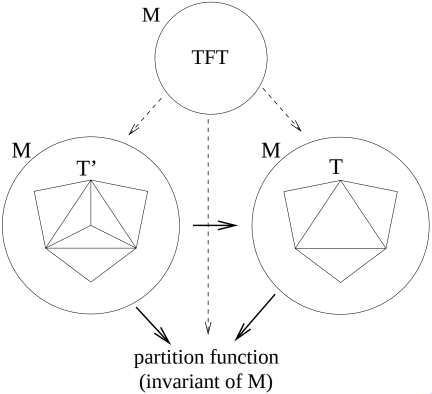
- space of fields $F_M = \Gamma(M, \mathfrak{F}_M) \ni \phi$,
- action $S_M(\phi) = \int_M L(\phi, \partial\phi, \partial^2\phi, \cdots)$, invariant under diffeomorphisms of M.

Quantum partition function:

$$Z_M = \int_{F_M} \mathcal{D}\phi \; e^{\frac{i}{\hbar} S_M(\phi)}$$

– a diffeomorphism invariant of M to be defined e.g. via perturbative (Feynman diagram) calculation as an asymptotic series at $\hbar \to 0$.

Simplicial program for TFTs: Given a TFT on a manifold M with space of fields F_M and action $S_M \in C^\infty(F_M)[[\hbar]]$, construct an **exact discretization** associating to a triangulation T of M a fin.dim. space F_T and a **local** action $S_T \in C^\infty(F_T)[[\hbar]]$, such that partition function Z_M and correlation functions can be obtained from (F_T, S_T) by fin.dim. integrals. Also, if T' is a subdivision of T, S_T is an effective action for $S_{T'}$.



Example of a TFT for which the exact discretization exists:
BF **theory:**

- Fields: $F_M = \mathfrak{g} \otimes \Omega^1(M) \oplus \mathfrak{g}^* \otimes \Omega^{n-2}(M)$,
  BV fields: $\mathcal{F}_M = \mathfrak{g} \otimes \Omega^\bullet(M)[1] \oplus \mathfrak{g}^* \otimes \Omega^\bullet(M)[n-2] \ \ni (A, B)$.
- Action: $S_M = \int_M \langle B \overset{\wedge}{,} dA + \frac{1}{2}[A \overset{\wedge}{,} A] \rangle$.

Realization of BF theory on a triangulation. (Exact discretization.)
Reference: P. Mnev, *Notes on simplicial BF theory*, Moscow Math. J 9.2 (2009): 371–410.
P. Mnev, *Discrete BF theory*, arXiv:0809.1160.

Fix T a triangulation of M.

- Fields:

$$\mathcal{F}_T = \mathfrak{g}\otimes C^\bullet(T)[1]\oplus\mathfrak{g}^*\otimes C_\bullet(T)[-2] \quad \ni (A = \sum_{\sigma\in T} A^\sigma e_\sigma, B = \sum_{\sigma\in T} B_\sigma e^\sigma)$$

with

$$\begin{aligned} A^\sigma &\in \mathfrak{g}, & B_\sigma &\in \mathfrak{g}^* \\ \mathrm{gh}\, A^\sigma &= 1-|\sigma|, & \mathrm{gh}\, B_\sigma &= -2+|\sigma| \end{aligned}$$

- Action: $S_T = \sum_{\sigma\in T} \bar{S}_\sigma(\{A^{\sigma'}\}_{\sigma'\subset\sigma}, B_\sigma; \hbar)$

Here $\bar{S}_\sigma$ – universal local building block, depending only on the dimension of σ.

Universal local building blocks

- For $\dim\sigma = 0$ a **point**, $\bar{S}_{\mathrm{pt}} = \langle B_0, \frac{1}{2}[A^0, A^0]\rangle$.
- For $\dim\sigma = 1$ an **interval**,

$$\bar{S}_{01} = \left\langle B_{01}, [A^{01}, \frac{A^0 + A^1}{2}] + \mathsf{F}(\mathrm{ad}_{A^{01}})(A^1 - A^0) \right\rangle - i\hbar \log \det{}_{\mathfrak{g}} \mathsf{G}(\mathrm{ad}_{A^{01}})$$

with $\mathsf{F}(x) = \frac{x}{2}\coth\frac{x}{2}$, $\mathsf{G}(x) = \frac{2}{x}\sinh\frac{x}{2}$.

- For $\dim\sigma \geq 2$,

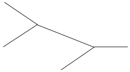
$$\bar{S}_\sigma =$$

$$\sum_{n\geq 1}\sum_T \sum_{\sigma_1,\ldots,\sigma_n\subset\sigma} \frac{1}{|\mathrm{Aut}(T)|} C(T)^\sigma_{\sigma_1\cdots\sigma_n} \langle B_\sigma, \mathrm{Jacobi}_{\mathfrak{g}}(T; A^{\sigma_1}, \cdots, A^{\sigma_k})\rangle -$$

$$-i\hbar \sum_{n\geq 2}\sum_L \sum_{\sigma_1,\ldots,\sigma_n\subset\sigma} \frac{1}{|\mathrm{Aut}(L)|} C(L)_{\sigma_1\cdots\sigma_n} \mathrm{Jacobi}_{\mathfrak{g}}(L; A^{\sigma_1}, \cdots, A^{\sigma_k})$$

Here T runs over rooted binary trees, L runs over connected trivalent 1-loop graphs. $C(T), C(L) \in \mathbb{Q}$ are **structure constants**.

Examples of structure constants.

$C($  $)^{\sigma}_{\sigma_1\sigma_2\sigma_3}$

$$= \begin{cases} \pm\frac{|\sigma_1|!\cdot|\sigma_2|!\cdot|\sigma_3|!}{(|\sigma_1|+|\sigma_2|+1)\cdot(|\sigma|+2)!} \\ 0 \end{cases}$$

depending on the combinatorics of the triple of faces $\sigma_1, \sigma_2, \sigma_3 \subset \sigma$.

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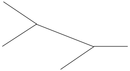
$C($  $)^{\sigma}_{\sigma_1\sigma_2\sigma_3} = \begin{cases} \pm\frac{|\sigma_1|!\cdot|\sigma_2|!\cdot|\sigma_3|!}{(|\sigma_1|+|\sigma_2|+1)\cdot(|\sigma|+2)!} \\ 0 \end{cases}$

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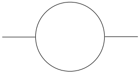
$C($  $)^{\sigma}_{\sigma_1\sigma_2} = \begin{cases} \pm\frac{1}{(|\sigma|+1)^2\cdot(|\sigma|+2)} \\ 0 \end{cases}$

where the nonzero structure constant corresponds to $\sigma_1 = \sigma_2$ an edge (1-simplex) of σ.

Effective action on cohomology Consider BV pushforward to
$\mathcal{F}_{H^\bullet} = \mathfrak{g} \otimes H^\bullet(M) \oplus \mathfrak{g}^* \otimes H_\bullet(M)$,
$S_{H^\bullet} = \langle B, \sum_{n\geq 2} \frac{1}{n!} l_n^{H^\bullet}(A,\ldots,A)\rangle - i\hbar \sum_{n\geq 2} \frac{1}{n!} q_n^{H^\bullet}(A,\ldots,A)$.

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Operations l_n are Massey brackets and encode the rational homotopy type of M; q_n correspond to the expansion of R-torsion near zero connection on the moduli space of flat connections on M. This invariant is stronger than rational homotopy type.

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Example:

1. $M = S^1$,

$$S_{H^\bullet} = \langle B_{(0)}, \frac{1}{2}[A^{(0)}, A^{(0)}]\rangle + \langle B_{(1)}, [A^{(0)}, A^{(1)}]\rangle - i\hbar \log \det_{\mathfrak{g}} \frac{\sinh \frac{\mathrm{ad}_{A^{(1)}}}{2}}{\frac{\mathrm{ad}_{A^{(1)}}}{2}}$$

2. M the Klein bottle,

$$S_{H^\bullet} = \langle B_{(0)}, \frac{1}{2}[A^{(0)}, A^{(0)}]\rangle + \langle B_{(1)}, [A^{(0)}, A^{(1)}]\rangle - i\hbar \log \det_{\mathfrak{g}} \frac{\tanh \frac{\mathrm{ad}_{A^{(1)}}}{2}}{\frac{\mathrm{ad}_{A^{(1)}}}{2}}$$

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Example:

1. $M = S^1$,

$$S_{H^\bullet} = \langle B_{(0)}, \frac{1}{2}[A^{(0)}, A^{(0)}]\rangle + \langle B_{(1)}, [A^{(0)}, A^{(1)}]\rangle - i\hbar \log \det{}_{\mathfrak{g}} \frac{\sinh \frac{\mathrm{ad}_{A^{(1)}}}{2}}{\frac{\mathrm{ad}_{A^{(1)}}}{2}}$$

2. M the Klein bottle,

$$S_{H^\bullet} = \langle B_{(0)}, \frac{1}{2}[A^{(0)}, A^{(0)}]\rangle + \langle B_{(1)}, [A^{(0)}, A^{(1)}]\rangle - i\hbar \log \det{}_{\mathfrak{g}} \frac{\tanh \frac{\mathrm{ad}_{A^{(1)}}}{2}}{\frac{\mathrm{ad}_{A^{(1)}}}{2}}$$

$S^1 \sim \text{Klein Bottle}$ rationally, but distinguished by quantum operations on cohomology.

One-dimensional simplicial Chern-Simons theory.

Reference: A. Alekseev, P. Mnev, *One-dimensional Chern-Simons theory,* Comm. Math. Phys. 307.1 (2011) 185–227.

Continuum theory on a circle. Fix $(\mathfrak{g}, \langle , \rangle)$ be a *quadratic* even-dimensional Lie algebra.

- Fields: A – a $\mathfrak{g}$-valued 1-form, ψ – an odd $\mathfrak{g}$-valued 0-form. The odd symplectic structure: $\omega = \int_{S^1} \langle \delta\psi \overset{\wedge}{,} \delta A \rangle$
- Action: $S(\psi, A) = \int_{S^1} \langle \psi \overset{\wedge}{,} d\psi + [A, \psi] \rangle$

BV pushforward to cochains of triangulated circle.
Denote T_N the triangulation of S^1 with N vertices. Discrete space of fields: cellular 0- and 1-cochains of T_N with values in $\mathfrak{g}$, with coordinates $\{\psi_k \in \Pi\mathfrak{g}, A_k \in \mathfrak{g}\}_{k=1}^N$ and odd symplectic form

$$\omega_{T_N} = \sum_{k=1}^{N} \left\langle \underbrace{\delta\left(\frac{\psi_k + \psi_{k+1}}{2}\right)}_{\tilde{\psi}_k}, \delta A_k \right\rangle$$

Explicit simplicial Chern-Simons action on cochains of triangulated circle:

$$S_{T_N} =$$

$$= -\frac{1}{2}\sum_{k=1}^{N}\left((\psi_k,\psi_{k+1}) + \frac{1}{3}(\psi_k,\mathrm{ad}_{A_k}\psi_k) + \frac{1}{3}(\psi_{k+1},\mathrm{ad}_{A_k}\psi_{k+1}) + \frac{1}{3}(\psi_k,\mathrm{ad}_{A_k}\psi_{k+1})\right) +$$

$$+\frac{1}{2}\sum_{k=1}^{N}(\psi_{k+1}-\psi_k,\left(\frac{1-R(\mathrm{ad}_{A_k})}{2}\left(\frac{1}{1+\mu_k(A')} - \frac{1}{1+R(\mathrm{ad}_{A_k})}\right)\frac{1-R(\mathrm{ad}_{A_k})}{2R(\mathrm{ad}_{A_k})} +\right.$$

$$\left. +(\mathrm{ad}_{A_k})^{-1} + \frac{1}{12}\mathrm{ad}_{A_k} - \frac{1}{2}\coth\frac{\mathrm{ad}_{A_k}}{2}\right)\circ(\psi_{k+1}-\psi_k))+$$

$$+\frac{1}{2}\sum_{k'=1}^{N}\sum_{k=k'+1}^{k'+N-1}(-1)^{k-k'}(\psi_{k+1}-\psi_k,\frac{1-R(\mathrm{ad}_{A_k})}{2}R(\mathrm{ad}_{A_{k-1}})\cdots R(\mathrm{ad}_{A_{k'}})\cdot$$

$$\cdot\frac{1}{1+\mu_{k'}(A')}\cdot\frac{1-R(\mathrm{ad}_{A_{k'}})}{2R(\mathrm{ad}_{A_{k'}})}\circ(\psi_{k'+1}-\psi_{k'}))+$$

$$+\hbar\,\frac{1}{2}\mathrm{tr}_{\mathfrak{g}}\log\left((1+\mu_\bullet(A'))\prod_{k=1}^{n}\left(\frac{1}{1+R(\mathrm{ad}_{A_k})}\cdot\frac{\sinh\frac{\mathrm{ad}_{A_k}}{2}}{\frac{\mathrm{ad}_{A_k}}{2}}\right)\right)$$

where

$$R(\mathcal{A}) = -\frac{\mathcal{A}^{-1}+\frac{1}{2}-\frac{1}{2}\coth\frac{\mathcal{A}}{2}}{\mathcal{A}^{-1}-\frac{1}{2}-\frac{1}{2}\coth\frac{\mathcal{A}}{2}},\quad \mu_k(A') = R(\mathrm{ad}_{A_{k-1}})R(\mathrm{ad}_{A_{k-2}})\cdots R(\mathrm{ad}_{A_{k+1}})R(\mathrm{ad}_{A_k})$$

Questions:

- Why such a long formula?
- It is not simplicially local (there are monomials involving distant simplices). How to disassemble the result into contributions of individual simplices?
- How to check quantum master equation for S_{T_N} explicitly?
- Simplicial aggregations should be given by finite-dimensional BV integrals; how to check that?

Introduce the **building block**

$$\zeta(\underbrace{\tilde{\psi}}_{\in \Pi\mathfrak{g}}, \underbrace{A}_{\in \mathfrak{g}}) = (i\hbar)^{-\frac{\dim \mathfrak{g}}{2}} \int_{\Pi\mathfrak{g}} D\lambda \, \exp\left(-\frac{1}{2\hbar}\langle\hat{\psi}, [A, \hat{\psi}]\rangle + \langle\lambda, \hat{\psi} - \tilde{\psi}\rangle\right) \in Cl(\mathfrak{g})$$

where $\{\hat{\psi}^a\}$ are generators of the Clifford algebra $Cl(\mathfrak{g})$, $\hat{\psi}^a\hat{\psi}^b + \hat{\psi}^b\hat{\psi}^a = \hbar\delta^{ab}$

Theorem (A.Alekseev, P.M.)

1. For a triangulated circle, $e^{\frac{i}{\hbar}S_{T_N}} = \mathrm{Str}_{Cl(\mathfrak{g})}\left(\zeta(\tilde{\psi}_N, A_N) * \cdots * \zeta(\tilde{\psi}_1, A_1)\right)$
2. The bulding block satisfies the *modified* quantum master equation

$$\hbar\Delta\zeta + \frac{1}{\hbar}\left[\frac{1}{6}\langle\hat{\psi}, [\hat{\psi}, \hat{\psi}]\rangle, \zeta\right]_{Cl(\mathfrak{g})} = 0$$

   where $\Delta = \frac{\partial}{\partial\tilde{\psi}}\frac{\partial}{\partial A}$.
3. Simplicial action on triangulated circle S_{T_N} satisfies the usual BV quantum master equation, $\Delta_{T_N} e^{\frac{i}{\hbar}S_{T_N}} = 0$, where $\Delta_{T_N} = \sum_k \frac{\partial}{\partial\tilde{\psi}_k}\frac{\partial}{\partial A_k}$.

Further developments

- Discrete BF theory and simplicial 1D Chern-Simons can be extended to triangulated manifolds with boundary, with Atiyah-Segal (functorial) cutting/pasting rule.
  References: A. Alekseev, P. Mnev, *One-dimensional Chern-Simons theory*, Comm. Math. Phys. 307.1 (2011) 185–227.
  A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Cellular BV-BFV-BF theory*, in preparation.
- Pushforward to cohomology in perturbative Chern-Simons theory - yields perturbative invariants of 3-manifolds without acyclicity condition on background local system.
  Reference: A. S. Cattaneo, P. Mnev, *Remarks on Chern-Simons invariants*, Comm. Math. Phys. 293.3 (2010) 803–836.
- Pushforward to cohomology in Poisson sigma model.
  Reference: F. Bonechi, A. S. Cattaneo, P. Mnev, *The Poisson sigma model on closed surfaces*, JHEP 2012.1 (2012) 1–27.

Pushforward to residual fields is made compatible with functorial cutting-pasting in the programme of
perturbative BV quantization on manifolds with boundary/corners,

References: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332.2 (2014) 535–603.
A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Perturbative quantum gauge theories on manifolds with boundary,* arXiv:1507.01221.

Short survey: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Perturbative BV theories with Segal-like gluing,* arXiv:1602.00741.