Hidden algebraic structure on cohomology of simplicial complexes, and TFT

Pavel Mnev

University of Zurich

Trinity College Dublin, February 4, 2013
Unimodular $L_\infty$ algebra associated to a simplicial complex

Background

Simplicial complex $T$

Simplicial cochains $C_0(T) \to \cdots \to C_{\text{top}}(T)$, $C_k(T) = \text{Span}\{k-\text{simplices}\}$, $d_k : C_k(T) \to C_{k+1}(T)$, $e_{\sigma} \mapsto \sum_{\sigma' \in T : \sigma \in \text{faces}(\sigma')} \pm e_{\sigma'}$

Cohomology $H^\bullet(T)$, $H_k(T) = \ker d_k / \text{im} d_{k-1}$ — a homotopy invariant of $T$
Simplicial complex $T$

Simplicial cochains $C^0(T) \to \cdots \to C^\text{top}(T)$,

$C^k(T) = \text{Span}\{k - \text{simplices}\}$,

$d_k : C^k(T) \to C^{k+1}(T)$, \quad $e_\sigma \quad \mapsto \quad \sum_{\sigma' \in T: \sigma \in \text{faces}(\sigma')} \pm e_{\sigma'}$

basis cochain
Simplicial complex $T$

\[ \downarrow \]

Simplicial cochains $C^0(T) \to \cdots \to C^{\text{top}}(T)$,

\[ C^k(T) = \text{Span}\{k \text{- simplices}\}, \]

\[ d_k : C^k(T) \to C^{k+1}(T), \quad \sum_{\sigma' \in T: \sigma \in \text{faces}(\sigma')} \pm e_{\sigma'} \]

\[ \downarrow \]

Cohomology $H^\bullet(T), \quad H^k(T) = \ker d_k / \text{im } d_{k-1}$

— a homotopy invariant of $T$
Cohomology carries a commutative ring structure, coming from (non-commutative) Alexander’s product for cochains.
Cohomology carries a commutative ring structure, coming from (non-commutative) Alexander’s product for cochains.

Massey operations on cohomology are a complete invariant of rational homotopy type in simply connected case (Quillen-Sullivan), i.e. rationalized homotopy groups $\mathbb{Q} \otimes \pi_k(T)$ can be recovered from them.
Cohomology carries a commutative ring structure, coming from (non-commutative) Alexander’s product for cochains.

Massey operations on cohomology are a complete invariant of rational homotopy type in simply connected case (Quillen-Sullivan), i.e. rationalized homotopy groups $\mathbb{Q} \otimes \pi_k(T)$ can be recovered from them.

Example of use: linking of Borromean rings is detected by a non-vanishing Massey operation on cohomology of the complement.

$m_3([\alpha], [\beta], [\gamma]) = [u \wedge \gamma + \alpha \wedge v] \in H^2$

where $[\alpha], [\beta], [\gamma] \in H^1$, $du = \alpha \wedge \beta$, $dv = \beta \wedge \gamma$. 
Another example: **nilmanifold**

\[ M = H_3(\mathbb{R})/H_3(\mathbb{Z}) \]

\[ = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \bigg/ \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \]

Denote

\[ \alpha = dx, \quad \beta = dy, \quad u = dz - y dx \in \Omega^1(M) \]

Important point: \( \alpha \wedge \beta = du \). The cohomology is spanned by classes

\[ [1], \ [\alpha], [\beta], \ [\alpha \wedge u], [\beta \wedge u], \ [\alpha \wedge \beta \wedge u] \]

degree 0 \quad \text{degree 1} \quad \text{degree 2} \quad \text{degree 3}

and

\[ m_3([\alpha], [\beta], [\beta]) = [u \wedge \beta] \in H^2(M) \]

is a non-trivial Massey operation.
Fix $\mathfrak{g}$ a unimodular Lie algebra (i.e. with $\text{tr}[x, \bullet] = 0$ for any $x \in \mathfrak{g}$).

**Main construction (P.M.)**

- Simplicial complex $T$  
  - Local formula  
  - Unimodular $L_\infty$ algebra structure on $\mathfrak{g} \otimes C^\bullet(T)$  
  - Homotopy transfer  
  - Unimodular $L_\infty$ algebra structure on $\mathfrak{g} \otimes H^\bullet(T)$
Fix \( g \) a unimodular Lie algebra (i.e. with \( \text{tr}[x, \bullet] = 0 \) for any \( x \in g \)).

**Main construction (P.M.)**

Simplicial complex \( T \)

\[
\begin{array}{c}
\text{local formula} \\
\downarrow \\
\text{Unimodular } L_\infty \text{ algebra structure on } g \otimes C^\bullet(T) \\
\downarrow \\
\text{homotopy transfer} \\
\text{Unimodular } L_\infty \text{ algebra structure on } g \otimes H^\bullet(T)
\end{array}
\]

**Main theorem (P.M.)**

Unimodular \( L_\infty \) algebra structure on \( g \otimes H^\bullet(T) \) (up to isomorphisms) is an invariant of \( T \) under simple homotopy equivalence.

horn filling collapse to a horn
Main construction (P.M.)

\[
\begin{align*}
\text{Simplicial complex } T & \quad \downarrow \text{local formula} \\
\text{Unimodular } L_\infty \text{ algebra structure on } \mathfrak{g} \otimes C^\bullet(T) & \quad \downarrow \text{homotopy transfer} \\
\text{Unimodular } L_\infty \text{ algebra structure on } \mathfrak{g} \otimes H^\bullet(T)
\end{align*}
\]

- Thom’s problem: lifting ring structure on \( H^\bullet(T) \) to a \textbf{commutative} product on cochains. Removing \( \mathfrak{g} \), we get a homotopy commutative algebra on \( C^\bullet(T) \). This is an improvement of Sullivan’s result with cDGA structure on cochains = \( \Omega_{\text{poly}}(T) \).

- \textbf{Local} formulae for Massey operations.

- Our invariant is strictly stronger than rational homotopy type.
A unimodular $L_\infty$ algebra is the following collection of data:

(a) a $\mathbb{Z}$-graded vector space $V^\bullet$,

(b) “classical operations” $l_n : \Lambda^n V \to V$, $n \geq 1$,

(c) “quantum operations” $q_n : \Lambda^n V \to \mathbb{R}$, $n \geq 1$,
Definition

A unimodular $L_\infty$ algebra is the following collection of data:

(a) a $\mathbb{Z}$-graded vector space $V^\bullet$,
(b) “classical operations” $l_n : \bigwedge^n V \to V$, $n \geq 1$,
(c) “quantum operations” $q_n : \bigwedge^n V \to \mathbb{R}$, $n \geq 1$,

subject to two sequences of quadratic relations:

1. $\sum_{r+s=n} \frac{1}{r!s!} l_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$, $n \geq 1$
   (anti-symmetrization over inputs implied),

2. $\frac{1}{n!} \text{Str} \ l_{n+1}(\bullet, \cdots, \bullet, -) +
   \sum_{r+s=n} \frac{1}{r!s!} q_{r+1}(\bullet, \cdots, \bullet, l_s(\bullet, \cdots, \bullet)) = 0$
A unimodular $L_\infty$ algebra is the following collection of data:

(a) a $\mathbb{Z}$-graded vector space $V^\bullet$,
(b) “classical operations” $l_n : \wedge^n V \to V$, $n \geq 1$,
(c) “quantum operations” $q_n : \wedge^n V \to \mathbb{R}$, $n \geq 1$,

subject to two sequences of quadratic relations:

1. \[
\sum_{r+s=n} \frac{1}{r!s!} l_{r+1}(\bullet, \ldots, \bullet, l_s(\bullet, \ldots, \bullet)) = 0, \quad n \geq 1
\]
   (anti-symmetrization over inputs implied),

2. \[
\frac{1}{n!} \text{Str} l_{n+1}(\bullet, \ldots, \bullet, -) + \\
+ \sum_{r+s=n} \frac{1}{r!s!} q_{r+1}(\bullet, \ldots, \bullet, l_s(\bullet, \ldots, \bullet)) = 0
\]

Note:

- First classical operation satisfies $(l_1)^2 = 0$, so $(V^\bullet, l_1)$ is a complex.
- A unimodular $L_\infty$ algebra is in particular an $L_\infty$ algebra (as introduced by Lada-Stasheff), by ignoring $q_n$.
- Unimodular Lie algebra is the same as unimodular $L_\infty$ algebra with $l_{\neq 2} = q_\bullet = 0$. 

A unimodular $L_\infty$ algebra is a graded vector space $V$ endowed with

- a vector field $Q$ on $V[1]$ of degree 1,
- a function $\rho$ on $V[1]$ of degree 0,

satisfying the following identities:

$$[Q, Q] = 0, \quad \text{div } Q = Q(\rho)$$
Homotopy transfer theorem (P.M.)

If \((V, \{l_n\}, \{q_n\})\) is a unimodular \(L_\infty\) algebra and \(V' \hookrightarrow V\) is a deformation retract of \((V, l_1)\), then

\(V'\) carries a unimodular \(L_\infty\) structure given by

\[ l'_n = \sum \Gamma_0 \frac{1}{|\text{Aut}(\Gamma_0)|} \quad : \wedge^n V' \to V' \]

\[ q'_n = \sum \Gamma_1 \frac{1}{|\text{Aut}(\Gamma_1)|} + \sum \Gamma_0 \frac{1}{|\text{Aut}(\Gamma_0)|} \quad : \wedge^n V' \to \mathbb{R} \]

where \(\Gamma_0\) runs over rooted trees with \(n\) leaves and \(\Gamma_1\) runs over 1-loop graphs with \(n\) leaves.
Homotopy transfer theorem (P.M.)

If \((V, \{l_n\}, \{q_n\})\) is a unimodular \(L_\infty\) algebra and \(V' \hookrightarrow V\) is a deformation retract of \((V, l_1)\), then 

1. \(V'\) carries a unimodular \(L_\infty\) structure given by

\[
\begin{align*}
    l_n' &= \sum \frac{1}{|\text{Aut}(\Gamma_0)|} \Gamma_0 : \wedge^n V' \to V' \\
    q_n' &= \sum \frac{1}{|\text{Aut}(\Gamma_1)|} \Gamma_1 + \sum \frac{1}{|\text{Aut}(\Gamma_0)|} \Gamma_0 : \wedge^n V' \to \mathbb{R}
\end{align*}
\]

where \(\Gamma_0\) runs over rooted trees with \(n\) leaves and \(\Gamma_1\) runs over 1-loop graphs with \(n\) leaves. **Decorations:**

<table>
<thead>
<tr>
<th>Decoration</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>leaf</td>
<td>(i : V' \hookrightarrow V)</td>
</tr>
<tr>
<td>edge</td>
<td>(-s : V \rightarrow V^{\bullet-1})</td>
</tr>
<tr>
<td>cycle</td>
<td>super-trace over (V)</td>
</tr>
<tr>
<td>root</td>
<td>((m+1))-valent vertex</td>
</tr>
<tr>
<td></td>
<td>(m)-valent (\circ)-vertex</td>
</tr>
<tr>
<td></td>
<td>(p : V \rightarrow V')</td>
</tr>
<tr>
<td></td>
<td>(l_m)</td>
</tr>
<tr>
<td></td>
<td>(q_m)</td>
</tr>
</tbody>
</table>

where \(s\) is a chain homotopy, \(l_1 s + s l_1 = \text{id} - i p\).
Homotopy transfer theorem (P.M.)

If \((V, \{l_n\}, \{q_n\})\) is a unimodular \(L_\infty\) algebra and \(V' \hookrightarrow V\) is a deformation retract of \((V, l_1)\), then

1. \(V'\) carries a unimodular \(L_\infty\) structure given by

\[
l'_n = \sum \frac{1}{|\text{Aut}(\Gamma_0)|} \Gamma_0 : \wedge^n V' \to V'
\]

\[
q'_n = \sum \frac{1}{|\text{Aut}(\Gamma_1)|} \Gamma_1 \circ \bigcirc + \sum \frac{1}{|\text{Aut}(\Gamma_0)|} \Gamma_0 : \wedge^n V' \to \mathbb{R}
\]

where \(\Gamma_0\) runs over rooted trees with \(n\) leaves and \(\Gamma_1\) runs over 1-loop graphs with \(n\) leaves. **Decorations:**

<table>
<thead>
<tr>
<th>leaf</th>
<th>edge</th>
<th>cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i : V' \hookrightarrow V)</td>
<td>(-s : V^\bullet \to V^\bullet^{-1})</td>
<td>super-trace over (V)</td>
</tr>
<tr>
<td>(\text{root})</td>
<td>((m + 1))-valent vertex</td>
<td>(m)-valent o-vertex</td>
</tr>
<tr>
<td></td>
<td>(l_m)</td>
<td>(q_m)</td>
</tr>
</tbody>
</table>

where \(s\) is a chain homotopy, \(l_1 s + s l_1 = \text{id} - i p\).

2. Algebra \((V', \{l'_n\}, \{q'_n\})\) changes by isomorphisms under changes of induction data \((i, p, s)\).
Algebraic structure on simplicial cochains

Locality of the algebraic structure on simplicial cochains

\[
\begin{align*}
\ell_n^T(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n}) &= \sum_{\sigma \in T : \sigma_1, \ldots, \sigma_n \in \text{faces}(\sigma)} \bar{\ell}_n^\sigma(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n})e_{\sigma} \\
q_n^T(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n}) &= \sum_{\sigma \in T : \sigma_1, \ldots, \sigma_n \in \text{faces}(\sigma)} \bar{q}_n^\sigma(X_{\sigma_1}e_{\sigma_1}, \cdots, X_{\sigma_n}e_{\sigma_n})
\end{align*}
\]

Notations: \(e_\sigma\) – basis cochain for a simplex \(\sigma\), \(X_\bullet \in \mathfrak{g}\), \(X e_\sigma := X \otimes e_\sigma\).

Here \(\bar{\ell}_n^\sigma : \wedge^n(\mathfrak{g} \otimes C^\bullet(T)) \to \mathfrak{g}\), \(\bar{q}_n^\sigma : \wedge^n(\mathfrak{g} \otimes C^\bullet(T)) \to \mathbb{R}\) are universal local building blocks, depending on dimension of \(\sigma\) only, not on combinatorics of \(T\).
Zero-dimensional simplex $\sigma = [A]$:

$\bar{l}_2(X e_A, Ye_A) = [X, Y]$, all other operations vanish.
Zero-dimensional simplex $\sigma = [A]$:  
\[ \bar{l}_2(Xe_A, Ye_A) = [X, Y], \text{ all other operations vanish.} \]

One-dimensional simplex $\sigma = [AB]$:  
\[ \bar{l}_{n+1}(X_1 e_{AB}, \cdots , X_n e_{AB}, Ye_B) = \frac{B_n}{n!} \sum_{\theta \in S_n} [X_{\theta_1}, \cdots , [X_{\theta_n}, Y] \cdots ] \]
\[ \bar{l}_{n+1}(X_1 e_{AB}, \cdots , X_n e_{AB}, Ye_A) = (-1)^{n+1} \frac{B_n}{n!} \sum_{\theta \in S_n} [X_{\theta_1}, \cdots , [X_{\theta_n}, Y] \cdots ] \]
\[ \bar{q}_n(X_1 e_{AB}, \cdots , X_n e_{AB}) = \frac{B_n}{n \cdot n!} \sum_{\theta \in S_n} \text{tr}_g [X_{\theta_1}, \cdots , [X_{\theta_n}, \bullet] \cdots ] \]

where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$ are Bernoulli numbers.
Higher-dimensional simplices, \( \sigma = \Delta^N, \ N \geq 2 \): \( \bar{l}_n, \bar{q}_n \) are given by a regularized homotopy transfer formula for transfer
\( g \otimes \Omega^\bullet(\Delta^N) \to g \otimes C^\bullet(\Delta^N) \)
Higher-dimensional simplices, $\sigma = \Delta^N$, $N \geq 2$: $\bar{l}_n, \bar{q}_n$ are given by a regularized homotopy transfer formula for transfer $g \otimes \Omega^\bullet(\Delta^N) \to g \otimes C^\bullet(\Delta^N)$, with

- $i =$ representation of cochains by Whitney elementary forms,
- $p =$ integration over faces,
- $s =$ Dupont’s chain homotopy operator.
Higher-dimensional simplices, $\sigma = \Delta^N$, $N \geq 2$: $\bar{l}_n$, $\bar{q}_n$ are given by a regularized homotopy transfer formula for transfer $g \otimes \Omega^\bullet(\Delta^N) \to g \otimes C^\bullet(\Delta^N)$, with

- $i =$ representation of cochains by Whitney elementary forms,
- $p =$ integration over faces,
- $s =$ Dupont’s chain homotopy operator.

$$\left\{ \begin{array}{c} \bar{l}_n^\sigma \\ \bar{q}_n^\sigma \end{array} \right\} (X_{\sigma_1} e_{\sigma_1}, \cdots, X_{\sigma_n} e_{\sigma_n}) = \sum_{\Gamma} C(\Gamma)_{\sigma_1 \cdots \sigma_n}^\sigma \text{Jacobi}_g(\Gamma; X_{\sigma_1}, \cdots, X_{\sigma_n})$$

where $\Gamma$ runs over binary rooted trees with $n$ leaves for $\bar{l}_n^\sigma$ and over trivalent 1-loop graphs with $n$ leaves for $\bar{q}_n^\sigma$;

$C(\Gamma)_{\sigma_1 \cdots \sigma_n}^\sigma \in \mathbb{R}$ are structure constants.
Higher-dimensional simplices, $\sigma = \Delta^N$, $N \geq 2$: $\bar{l}_n$, $\bar{q}_n$ are given by a regularized homotopy transfer formula for transfer $g \otimes \Omega^\bullet(\Delta^N) \to g \otimes C^\bullet(\Delta^N)$, with

- $i = \text{representation of cochains by Whitney elementary forms},$
- $p = \text{integration over faces},$
- $s = \text{Dupont’s chain homotopy operator}.$

$$\begin{align*}
\bar{l}_n^\sigma & \left\{ X_{\sigma_1} e_{\sigma_1}, \cdots, X_{\sigma_n} e_{\sigma_n} \right\} = \sum_{\Gamma} C(\Gamma)^\sigma_{\sigma_1 \cdots \sigma_n} \text{Jacobi}_g(\Gamma; X_{\sigma_1}, \cdots, X_{\sigma_n})
\end{align*}$$

where $\Gamma$ runs over binary rooted trees with $n$ leaves for $\bar{l}_n^\sigma$ and over trivalent 1-loop graphs with $n$ leaves for $\bar{q}_n^\sigma$;

$C(\Gamma)^\sigma_{\sigma_1 \cdots \sigma_n} \in \mathbb{R}$ are structure constants.

There are explicit formulae for structure constants for small $n$. 
Summary: logic of the construction

building blocks $\bar{l}_n, \bar{q}_n$ on $\Delta^N$

\[\downarrow\text{combinatorics of } T\]

algebraic structure on cochains

\[\downarrow\text{homotopy transfer}\]

algebraic structure on cohomology

Operations $\bar{l}_n$ on $g \otimes H^\ast(T)$ are Massey brackets on cohomology and are a complete invariant of rational homotopy type in the simply-connected case.

Operations $\bar{q}_n$ on $g \otimes H^\ast(T)$ give a version of Reidemeister torsion of $T$.

Construction above yields new local combinatorial formulae for Massey brackets (in other words: Massey brackets lift to a local algebraic structure on simplicial cochains).
Summary: logic of the construction

building blocks $\bar{l}_n, \bar{q}_n$ on $\Delta^N$

\[\downarrow\text{combinatorics of } T\]

algebraic structure on cochains

\[\downarrow\text{homotopy transfer}\]

algebraic structure on cohomology

- Operations $l_n$ on $g \otimes H^\bullet(T)$ are **Massey brackets** on cohomology and are a complete invariant of **rational homotopy type** in simply-connected case.

- Operations $q_n$ on $g \otimes H^\bullet(T)$ give a version of **Reidemeister torsion** of $T$.

- Construction above yields new local combinatorial formulae for Massey brackets (in other words: Massey brackets lift to a local algebraic structure on simplicial cochains).
Example: for a circle and a Klein bottle, $H^\bullet(S^1) \simeq H^\bullet(KB)$ as rings, but $g \otimes H^\bullet(S^1) \not\simeq g \otimes H^\bullet(KB)$ as unimodular $L_\infty$ algebras (distinguished by quantum operations).

$$e\sum_n \frac{1}{n!} q_n(X \otimes \epsilon, \ldots X \otimes \epsilon) =$$

\[
\begin{array}{|c|c|}
\hline
\det_g \left( \frac{\sinh \frac{\text{ad} X}{2}}{\frac{\text{ad} X}{2}} \right) & \det_g \left( \frac{\text{ad} X}{2} \cdot \coth \frac{\text{ad} X}{2} \right)^{-1} \\
\text{for } S^1 & \text{for Klein bottle} \\
\hline
\end{array}
\]

where $\epsilon \in H^1$ – generator, $X \in g$ – variable.
Example: Massey bracket on the nilmanifold, combinatorial calculation

Triangulation of the nilmanifold:

- **one 0-simplex**: $A = B = C = D = A' = B' = C' = D'$
- **seven 1-simplices**:
  - $AD = BC = A'D' = B'C'$,
  - $AA' = BB' = CC' = DD'$,
  - $AB = DC = D'B'$,
  - $AC = A'B' = D'C'$,
  - $AB' = DC'$,
  - $AD' = BC'$,
  - $AC'$
- **twelve 2-simplices**:
  - $AA'B' = DD'C'$,
  - $AB'B = DC'C$,
  - $AA'D' = BB'C'$,
  - $AD'D = BC'C$,
  - $ACD = AB'D'$,
  - $ABC = D'B'C'$,
  - $AB'D'$,
  - $AC'D'$,
  - $ACC'$,
  - $ABC$
- **six 3-simplices**:
  - $AA'B'D'$,
  - $AB'C'D'$,
  - $ADC'D'$,
  - $ABB'C'$,
  - $ABCC'$,
  - $ACDC'$

Massey bracket on $H_1$:

$\text{Massey bracket on } H_1 = l_3(X \otimes \alpha, Y \otimes \beta, Z \otimes \beta) = 1$ 

$T^2 l_T^2 X \otimes \alpha Y \otimes \beta Z \otimes \beta - s_T + 1/6 T^3 X \otimes \alpha Y \otimes \beta Z \otimes \beta + \text{permutations of inputs} = (\text{Massey bracket } X, Y, Z) \otimes \eta \in g \otimes H_2(T)$

where $s_T = d \lor (dd \lor + d \lor d)$; $\alpha = e_{AC} + e_{AD} + e_{AC'} + e_{AD'}$, $\beta = e_{AA'} + e_{AB'} + e_{AC'} + e_{AD'}$; representatives of cohomology classes $[\alpha], [\beta]$ in simplicial cochains.
Example: Massey bracket on the nilmanifold, combinatorial calculation

**Triangulation of the nilmanifold:**

- **one 0-simplex:** $A = B = C = D = A' = B' = C' = D'$
- **seven 1-simplices:** $AD = BC = A'D' = B'C'$,
  $AA' = BB' = CC' = DD'$, $AB = DC = D'B'$,
  $AC = A'B' = D'C'$, $AB' = DC'$, $AD' = BC'$, $AC'$
- **twelve 2-simplices:** $A'A'B' = DD'C'$, $AB'B = DC'C$,
  $AA'D' = BB'C'$, $AD'D = BC'C$, $ACD = AB'D'$,
  $ABC = D'B'C'$, $AB'D'$, $AC'D'$, $ACC'$, $ABC'$
- **six 3-simplices:** $A'A'B'D'$, $AB'C'D'$,
  $ADC'D'$, $ABB'C'$, $ABCC'$, $ACDC'$

**Massey bracket on $H^1$:**

\[
l_3(X \otimes [\alpha], Y \otimes [\beta], Z \otimes [\beta]) =
\]

\[
= \frac{1}{2} \left( X \otimes \alpha \right) - s^T_l \left( Y \otimes \beta \right) + \frac{1}{6} \left( Y \otimes \beta \right) \left( Z \otimes \beta \right)
+ \text{permutations of inputs}
\]

\[
= ([[X, Y], Z] + [[X, Z], Y]) \otimes [\eta] \in g \otimes H^2(T)
\]

where $s^T = d^\vee/(dd^\vee + d^\vee d)$;

$\alpha = e_{AC} + e_{AD} + e_{AC'} + e_{AD'}$, $\beta = e_{AA'} + e_{AB'} + e_{AC'} + e_{AD'}$

- representatives of cohomology classes $[\alpha], [\beta]$ in simplicial cochains.
**Simplicial program for TFTs:** Given a TFT on a manifold $M$ with space of fields $F_M$ and action $S_M \in C^\infty(F_M)[[\hbar]]$, construct an **exact discretization** associating to a triangulation $T$ of $M$ a fin.dim. space $F_T$ and a **local** action $S_T \in C^\infty(F_T)[[\hbar]]$, such that partition function $Z_M$ and correlation functions can be obtained from $(F_T, S_T)$ by fin.dim. integrals. Also, if $T'$ is a subdivision of $T$, $S_T$ is an effective action for $S_{T'}$. 
Example of a TFT for which the exact discretization exists:

**BF theory:**

- **fields:** $F_M = \underbrace{\mathfrak{g} \otimes \Omega^1(M)}_{A} \oplus \underbrace{\mathfrak{g}^* \otimes \Omega^{\dim M-2}(M)}_{B}$,

- **action:** $S_M = \int_M \langle B, dA + A \wedge A \rangle$,

- **equations of motion:** $dA + A \wedge A = 0$, $d_A B = 0$. 
### Algebra – TFT dictionary

<table>
<thead>
<tr>
<th>De Rham algebra $\mathfrak{g} \otimes \Omega^\bullet(M)$ (as a dg Lie algebra)</th>
<th>$BF$ theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unimodular $L_\infty$ algebra $(V, {l_n}, {q_n})$</td>
<td>$BF_\infty$ theory, $F = V[1] \oplus V^*[-2]$, $S = \sum_n \frac{1}{n!} \langle B, l_n(A, \cdots, A) \rangle + \hbar \sum_n \frac{1}{n!} q_n(A, \cdots, A)$</td>
</tr>
<tr>
<td>Quadratic relations on operations</td>
<td>Batalin-Vilkoviski master equation $\Delta e^{S/\hbar} = 0$</td>
</tr>
<tr>
<td>Homotopy transfer $V \to V'$</td>
<td>Effective action $e^{S'/\hbar} = \int_{L \subset F''} e^{S/\hbar}$, $F = F' \oplus F''$</td>
</tr>
<tr>
<td>Choice of chain homotopy $s$</td>
<td>Gauge-fixing (choice of Lagrangian $L \subset F''$)</td>
</tr>
</tbody>
</table>
**Goal:**

- Construct other simplicial TFTs, in particular simplicial Chern-Simons theory.
- Explore applications to invariants of manifolds and (generalized) knots, consistent with gluing-cutting.
Goal:

- Construct other simplicial TFTs, in particular simplicial Chern-Simons theory.
- Explore applications to invariants of manifolds and (generalized) knots, consistent with gluing-cutting.

Steps:

- Construct simplicial 1-dimensional Chern-Simons theory as Atiyah’s TFT on triangulated 1-cobordisms (complete, with Anton Alekseev).
- Construct a finite-dimensional algebraic model of 3-dimensional Chern-Simons theory; study effective action induced on de Rham cohomology and corresponding 3-manifold invariants (complete, with Alberto Cattaneo).
- Extend cohomological Batalin-Vilkovisky formalism for treating gauge symmetry of TFTs to allow spacetime manifolds with boundary or corners in a way consistent with gluing (complete, with Alberto Cattaneo and Nicolai Reshetikhin).
- Construct the quantization of TFTs on manifolds with boundary in BV formalism by perturbative path integral (in progress).
- Extend previous step to manifolds with corners (in progress).
References:


