Batalin-Vilkovisky effective actions and cutting-gluing

Pavel Mnev

University of Notre Dame

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Plan

1. Batalin-Vilkovisky formalism, effective actions
2. BV on manifolds with boundary ("BV-BFV")
3. BF-like theories and configuration space integrals
4. 2d Yang-Mills with corners
5. Chern-Simons on a cylinder and WZW

Batalin-Vilkovisky formalism

Classical Batalin-Vilkovisky formalism

Data:

- $\mathcal{F}$ space of fields
- ω odd symplectic form
- S action

Condition: $\{S,S\}=0$ ("classical master equation").
$\Rightarrow \quad Q=\{S,\bullet\}$ satisfies $Q^2=0$.

Quantization

$$\boxed{Z=\int_{\mathcal{L}\subset\mathcal{F}} e^{\frac{i}{\hbar}S}}$$ where
$\mathcal{L}\subset\mathcal{F}$ gauge-fixing Lagrangian;
S should satisfy the "quantum master equation":
$\Delta e^{\frac{i}{\hbar}S}=0 \quad \Leftrightarrow \frac{1}{2}\{S,S\}-i\hbar\Delta S=0.$

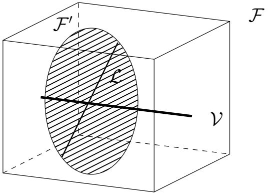
Observables: $\langle\mathcal{O}\rangle=\int_{\mathcal{L}\subset\mathcal{F}}\mathcal{O}e^{\frac{i}{\hbar}S}$ with $(Q-i\hbar\Delta)\mathcal{O}=0$.

Effective BV actions

Assume

- $\mathcal{F} = \mathcal{V} \times \mathcal{F}'$, $\mathcal{V}$= "slow (residual) fields";
- $\mathcal{L} \subset \mathcal{F}'$ Lagrangian.

Consider the fiber integral $Z = e^{\frac{i}{\hbar} S^{\mathrm{eff}}} = \int_{\mathcal{L} \subset \mathcal{F}'} e^{\frac{i}{\hbar} S}$.
S^{eff} – the "effective BV action" on $\mathcal{V}$.



Properties:

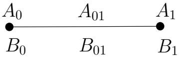
- $\Delta Z = 0$ (i.e. S^{eff} satisfies QME);
- Under a deformation of $\mathcal{L}$, $Z \mapsto Z + \Delta(\cdots)$.

Effective BV action and homotopy transfer of L_∞ algebras: example

Reference: P.M. "Notes on simplicial BF theory," arXiv:hep-th/0610326; "Discrete BF theory," arXiv:0809.1160

Example: $I=[0,1]$ interval, $\mathfrak{g}$ a Lie algebra.

- $\mathcal{F}=\Omega^\bullet(I,\mathfrak{g})[1]\oplus\Omega^\bullet(I,\mathfrak{g}^*)[-1]\ni(A,B)$, $\omega=\int\langle\delta A,\delta B\rangle$
- $S=\int_I\langle B,dA+\frac{1}{2}[A,A]\rangle$ ("BF theory on I")
- $\mathcal{V}=C^\bullet(I,\mathfrak{g})[1]\oplus C_\bullet(I,\mathfrak{g}^*)[-2]$
- Gauge-fixing corresponds to retraction $\Omega^\bullet(I)\overset{(i,p,K)}{\rightsquigarrow}C^\bullet(I)$.
- $S^{\mathrm{eff}}=\langle B_0,\frac{1}{2}[A_0,A_0]\rangle+\langle B_1,\frac{1}{2}[A_1,A_1]\rangle+$
  $+\left\langle B_{01},\left(\frac{\mathrm{ad}_{A_{01}}}{2}\coth\frac{\mathrm{ad}_{A_{01}}}{2}\right)\circ(A_1-A_0)\right\rangle-i\hbar\,\log\det_{\mathfrak{g}}\frac{\sinh\frac{\mathrm{ad}_{A_{01}}}{2}}{\frac{\mathrm{ad}_{A_{01}}}{2}}$
  – from Feynman diagram computation;
  propagator= chain contraction K



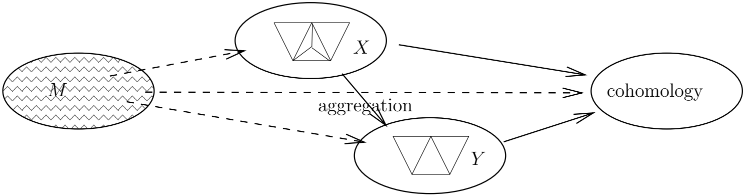
- $S\mapsto S^{\mathrm{eff}}$ corresponds to the homotopy transfer of (quantum) L_∞ algebras $\Omega^\bullet(I,\mathfrak{g}),d,\wedge\to C^\bullet(I,\mathfrak{g}),\{l_n,q_n\}$

Example cont'd

Generalization: $I \to M$ – a manifold; X – triangulation.

$$\begin{matrix} \mathcal{F} = \Omega^\bullet(M,\mathfrak{g})[1] \oplus \Omega^\bullet(M,\mathfrak{g}^*)[n-2] \\ S = \int_M \langle B, dA + \frac{1}{2}[A,A]\rangle \end{matrix} \longrightarrow \begin{matrix} \mathcal{V}_X = C^\bullet(X,\mathfrak{g})[1] \oplus C_\bullet(X,\mathfrak{g}^*)[-2] \\ S_X^{\mathrm{eff}} \end{matrix}$$

Poset of realizations of residual fields:



Classical BV-BFV

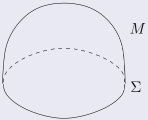
BV integrals + cutting-gluing = BV-BFV formalism

Reference: A. S. Cattaneo, P.M., N. Reshetikhin, "Classical BV theories on manifolds with boundary," arXiv:1201.0290;
"Perturbative quantum gauge theories on manifolds with boundary," arXiv:1507.01221

Classical BV for manifolds with boundary

n-manifold M $\quad\mapsto\quad$ $(\mathcal{F}_M, Q_M, \omega_M, S_M, \pi : \mathcal{F}_M \to \mathcal{F}_\Sigma)$

$(n-1)$-manifold $\Sigma = \partial M$ $\quad\mapsto\quad$ $(\mathcal{F}_\Sigma, Q_\Sigma, \omega_\Sigma = \delta\alpha_\Sigma, S_\Sigma)$



Relations: $Q^2 = 0$, $\pi_* Q_M = Q_\Sigma$, $Q_\Sigma = \{S_\Sigma, -\}_{\omega_\Sigma}$,

$$\boxed{\iota_{Q_M}\omega_M = \delta S_M + \pi^*\alpha_\Sigma} \quad \Rightarrow L_{Q_M} S_M = \pi^*(2S_\Sigma - \iota_{Q_\Sigma}\alpha_\Sigma)$$

Gluing $M_1 \cup_\Sigma M_2$ $\quad\mapsto\quad$ fiber product $\mathcal{F}_{M_1} \times_{\mathcal{F}_\Sigma} \mathcal{F}_{M_2}$

Classical BV-BFV: example

Example: abelian Chern-Simons

- M = 3-manifold with boundary Σ
- $\mathcal{F}_M = \Omega^\bullet(M)[1] \ni A$, $\omega_M = \frac{1}{2}\int_M \delta A \wedge \delta A$, $S_M = \frac{1}{2}\int_M A \wedge dA$, $Q_M A = dA$
- $\mathcal{F}_\Sigma = \Omega^\bullet(\Sigma)[1] \ni A_\Sigma$, $\alpha_\Sigma = \int_\Sigma A_\Sigma \wedge \delta A_\Sigma$, $S_\Sigma = \frac{1}{2}\int_\Sigma A_\Sigma \wedge dA_\Sigma$, $\pi : A \mapsto \iota^* A$ with $\iota : \Sigma \hookrightarrow M$.

Reduction (moduli spaces of dg manifolds):

$$\begin{array}{ccc} \mathcal{M}_M = \frac{\mathrm{zero}(Q_M)}{\sim} = H^\bullet(M)[1] & & \mathrm{deg} = +1 \;\; \mathrm{Poisson} \\ \downarrow \pi_* & & \\ \mathcal{M}_\Sigma = \frac{\mathrm{zero}(Q_\Sigma)}{\sim} = H^\bullet(\Sigma)[1] & & \mathrm{deg} = 0 \;\; \mathrm{symplectic} \end{array}$$

$\mathrm{im}(\pi_*) \subset \mathcal{M}_\Sigma$ – "evolution" Lagrangian.

Quantum BV-BFV

Quantum BV for manifolds with boundary

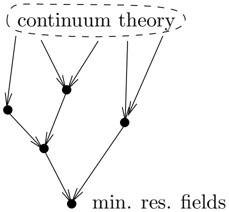
$(n-1)$-manifold Σ $\mapsto$ $(\mathcal{H}_\Sigma, \Omega_\Sigma)$ space of states

n-manifold M with $\partial M = \Sigma$ $\mapsto$ $(\mathcal{V}_M, \omega_\mathcal{V})$ residual fields
$Z_M \in \mathcal{H}_\Sigma \otimes \mathrm{Fun}(\mathcal{V}_M)$

(Modified) **quantum master equation:** $(\frac{i}{\hbar}\Omega_\Sigma - i\hbar\Delta_\mathcal{V})Z_M = 0.$

Gluing $M_1 \cup_\Sigma M_2$ $\mapsto$ $\langle Z_{M_1}, Z_{M_2}\rangle_{\mathcal{H}_\Sigma}$

Poset of realizations of $\mathcal{V}_M$:



Arrows $\mathcal{V}_1 \succ \mathcal{V}_2$ $\mapsto$ fiber BV integrals $Z^{\mathcal{V}_2} = \int^{BV} Z^{\mathcal{V}_1}$

Quantization – rough idea

- Fix a Lagrangian polarization $p\colon \mathcal{F}_\Sigma \to \mathcal{B}$ with α_Σ vanishing along fibers.
- $\mathcal{H}_\Sigma = \mathrm{Fun}(\mathcal{B})$, $\Omega_\Sigma = \widehat{S}_\Sigma$
- $\mathcal{Y}$= fiber of $\mathcal{F}_M \xrightarrow{\pi} \mathcal{F}_\Sigma \xrightarrow{p} \mathcal{B}$. Split it as $\mathcal{Y} = \mathcal{V} \times \mathcal{Y}'$.
- $$\boxed{Z(b, \phi_{\mathrm{res}}) = \int_{\mathcal{L}\subset\mathcal{Y}'} \mathcal{D}\phi_{\mathrm{fl}}\ e^{\frac{i}{\hbar}S(b,\phi_{\mathrm{res}},\phi_{\mathrm{fl}})}}, \qquad b \in \mathcal{B},\ \phi_{\mathrm{res}} \in \mathcal{V}$$

Example: AKSZ theories

Example: "BF-like" AKSZ theories.

- $\mathcal{F} = \Omega^\bullet(M) \otimes V \oplus \Omega^\bullet(M) \otimes V^*[n-1] \ni (A, B)$;
  V – a graded vector space of coefficients
- $\boxed{S = \int_M \langle B, dA \rangle + f(A, B)}$;
  $f \in \mathrm{Fun}(T^*[n-1]V)$ – AKSZ potential, satisfying $\{f, f\}_V = 0$.
- Boundary data: same formulas, replacing $M \to \Sigma$.

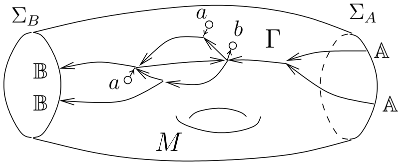
Example: AKSZ theories – quantization

Quantization:

- $\partial M = \Sigma_A \sqcup \Sigma_B$ – with A-fixed and B-fixed polarizations.
- $\mathcal{H}_{\Sigma_A} = \mathrm{Fun}(\mathbb{A}) = \{\sum_k \int_{\mathrm{Conf}_k(\Sigma_A)} \Psi_k(x_1, \ldots, x_k)\mathbb{A}(x_1)\cdots\mathbb{A}(x_k)\}$, $\mathcal{H}_{\Sigma_B}$ – similar
- $$\begin{array}{c} \mathcal{Y} = \Omega^\bullet(M, \Sigma_A) \otimes V \oplus \Omega^\bullet(M, \Sigma_B) \otimes V^*[n-1] \\ \downarrow \\ \mathcal{V} = H^\bullet(M, \Sigma_A) \otimes V \oplus H^\bullet(M, \Sigma_B) \otimes V^*[n-1] \quad \ni (a, b) \end{array}$$

Example: AKSZ theories – quantization, partition function

$$Z_M(\mathbb{A},\mathbb{B}\,|\,a,b) =$$
$$\tau(M,\Sigma_A)^{-\mathrm{sdim}\,V}\sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\int_{\mathrm{Conf}_\Gamma(M)} \omega_\Gamma(\mathbb{A},\mathbb{B}\,|\,a,b)$$



$$\omega_\Gamma = \bigwedge_{(v_1 v_2)\in E} \pi^*_{v_1 v_2}\eta \cdot \bigwedge_{v\in V_A} \pi^*_v \mathbb{A} \cdot \bigwedge_{v\in V_B} \pi^*_v \mathbb{B} \cdot \bigwedge_{v\in V_{\mathrm{int}}} \pi^*_v a^{\mathrm{val}_a(v)} \pi^*_v b^{\mathrm{val}_b(v)} \cdot \mathrm{Jacobi}_\Gamma(f)$$

$\eta \in \Omega^{n-1}(\mathrm{Conf}_2(M))$ – propagator, τ – Ray-Singer torsion

Example: AKSZ theories – quantization, propagator

Propagator

Propagator $\eta \in \Omega^{n-1}(\mathrm{Conf}_2(M))$ – integral kernel of a chain contraction

$$\begin{array}{cccc} K: & \Omega^\bullet(M,\Sigma_A) & \to & \Omega^{\bullet-1}(M,\Sigma_A) \\ & \alpha & \mapsto & \int_{M\ni x_2} \eta(x_1,x_2)\alpha(x_2) \end{array}$$

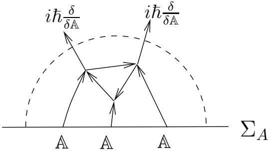
Properties:

- $d\eta = \sum_a \chi_a \otimes \chi^a$ with $\{\chi_a\}$ – basis in $H^\bullet(M,\Sigma_A)$
- $\lim_{\epsilon\to 0} \int_{S_\epsilon^{n-1}(y)\ni x} \eta(x,y) = 1$
- $\eta(x,y) = 0$ if $x \in \Sigma_A$
- $\eta(x,y) = 0$ if $y \in \Sigma_B$

Example: AKSZ theories – quantization, cont'd

Differential $\Omega_{\mathbb{A}}$ on $\mathcal{H}_{\Sigma_A}$ is built from codim=1 strata of $\mathrm{Conf}_\Gamma(M)$ corresponding to collapses of points near Σ_A:

$$\Omega_{\mathbb{A}} = i\hbar \int_{\Sigma_A} d\mathbb{A} \frac{\delta}{\delta \mathbb{A}} + \sum$$



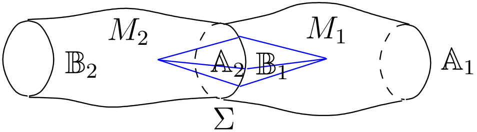
$\Omega_{\mathbb{B}}$ on $\mathcal{H}_{\Sigma_B}$ – similar (arrows reversed).

Properties:

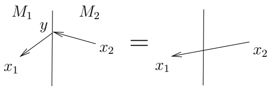
- $\Omega_{\mathbb{A}}^2 = 0$, $\Omega_{\mathbb{B}}^2 = 0$ (from Stokes' theorem on Conf)
- mQME: $\boxed{\left(\Omega_{\mathbb{A}} + \Omega_{\mathbb{B}} + \hbar^2 \Delta_{\mathcal{V}}\right) Z_M = 0}$ (Stokes' thm)

Example: AKSZ theories – quantization, cont'd

- Gluing holds: $Z_{M_1 \cup_\Sigma M_2} =$
$\int \mathcal{D}\mathbb{B}_1 \mathcal{D}\mathbb{A}_2 \;\; Z_{M_1}(\mathbb{A}_1, \mathbb{B}_1; a_1, b_1) e^{-\frac{i}{\hbar}\int_\Sigma \langle \mathbb{B}_1, \mathbb{A}_2 \rangle} Z_{M_2}(\mathbb{A}_2, \mathbb{B}_2; a_2, b_2)$



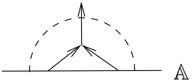
– due to a gluing formula for propagators



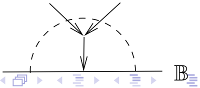
and Feynman graphs (and torsions)

Example: non-abelian BF **theory.** $V = \mathfrak{g}[1]$, $f(A, B) = \frac{1}{2}\langle B, [A, A]\rangle$.

$$\Omega_{\mathbb{A}} = \int_{\Sigma_A} \left\langle d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}], i\hbar \frac{\delta}{\delta \mathbb{A}} \right\rangle$$



$$\Omega_{\mathbb{B}} = \int_{\Sigma_B} \left\langle d\mathbb{B}, i\hbar \frac{\partial}{\partial \mathbb{B}} \right\rangle + \frac{1}{2}\left\langle \mathbb{B}, \left[i\hbar \frac{\delta}{\delta \mathbb{B}}, i\hbar \frac{\delta}{\delta \mathbb{B}} \right] \right\rangle$$



Example: AKSZ theories – Poisson sigma model

Example: 2d Poisson sigma model.

- M – surface with boundary
- $V = W^*[1]$, π – Poisson bivector on W
- $S = \int_M B^i dA_i + \frac{1}{2}\pi^{ij}(B)A_iA_j$,

  $(A,B) \in \Omega^\bullet(M,W^*)[1] \oplus \Omega^\bullet(M,W)$
- $\Omega^{\mathbb{B}}_{S^1}$ – standard-ordering quantization $\mathbb{A} \to i\hbar\frac{\delta}{\delta\mathbb{B}}$ of

$$\widetilde{S}_{S^1} = \oint_{S^1} \mathbb{B}^i d\mathbb{A}_i + \frac{1}{2}\Pi^{ij}(\mathbb{B})\mathbb{A}_i\mathbb{A}_j$$

  where $\Pi^{ij} = \dfrac{x^i *_\hbar x^j - x^j *_\hbar x^i}{i\hbar} \quad \in C^\infty(W)[[\hbar]]$,

  $*_\hbar$ – Kontsevich's star-product
- $\Omega^{\mathbb{A}}_{S^1}$ – quantization of $\widetilde{S}_{S^1}$ with $\mathbb{B} \to i\hbar\frac{\delta}{\delta\mathbb{A}}$

2d Yang-Mills

Action $\boxed{S = \int_M \left\langle B, dA + \frac{1}{2}[A, A] \right\rangle + \mu\,(B, B)}$

$(A, B) \in \Omega^\bullet(M, \mathfrak{g})[1] \oplus \Omega^\bullet(M, \mathfrak{g}^*)$;

$\mathfrak{g} = \mathrm{Lie}(G)$ with Killing form $(,)$; μ – area form.

Boundary structure – same as in BF.

$$\mathcal{H}^{\mathbb{A}}_{S^1} = \mathrm{Fun}(\Omega^\bullet(S^1, \mathfrak{g})[1]), \Omega^{\mathbb{A}}_{S^1} \quad \underset{H^0_\Omega}{\rightsquigarrow} \quad L^2(G)^G$$

2d Yang-Mills, cont'd

Problem: obtain a closed formula for the partition function on a closed surface M from a direct perturbative (Feynman diagram) computation.

Idea: Cut M into "building blocks" M_i admitting a convenient gauge-fixing, compute Z_{M_i} via Feynman diagrams, reassemble Z_M via BV-BFV gluing formula.

Theorem, Iraso-P.M.

For M a surface with $\partial M = \underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n}$ in $\mathbb{A}$-polarization,

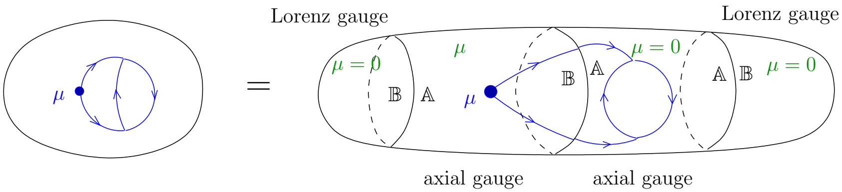
$$\left[\int_{\text{res. fields}} Z^{\text{BV}-\text{BFV}}(M)\right] = \sum_{R \text{ irrep of } G} (\dim R)^{\chi(M)} e^{-\frac{i\hbar \text{Area}(M)}{2} \cdot C_2(R)} |R\rangle^{\otimes n}$$

$[\cdots]$ = class in H^0_Ω; $\{|R\rangle\}$ – basis of characters in $L^2(G)^G$.

R.h.s. = non-perturbative answer by Migdal-Witten.

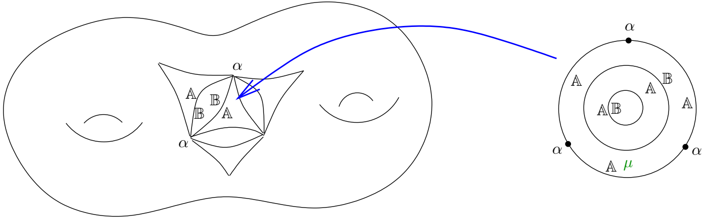
Reference: R. Iraso, P.M., "Two-dimensional Yang–Mills theory on surfaces with corners in Batalin–Vilkovisky formalism," arXiv:1806.04172

2d Yang-Mills: cutting into building blocks



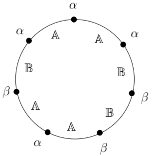
$0, 1$-loop diagrams on pieces assemble into multi-loop diagrams on M

For genus ≥ 2, need to cut into building blocks with corners:



2d Yang-Mills: quantization with corners

Can extend BV-BFV quantization to a picture with arcs and corners carrying polarizations.



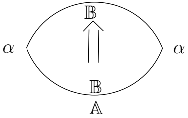
- arcs $\rightarrow$ $\mathbb{A}$ or $\mathbb{B}$-polarization
- corners (vertices) $\rightarrow$ α or β-polarization
- $\mathcal{H}^{\alpha}_{pt} = \mathrm{Fun}(\alpha) = \wedge^{\bullet}\mathfrak{g}^*$ with $\Omega = i\hbar\, d_{\mathrm{CE}}$;
  $\mathcal{H}^{\beta}_{pt} = \mathrm{Fun}(\beta) = U_\hbar\mathfrak{g}$ with $\Omega = 0$ – dg algebras
  – quantization of $(\mathcal{F}_{pt} = \mathfrak{g}[1] \oplus \mathfrak{g}^*,\ S_{pt} = \frac{1}{2}\langle\beta, [\alpha, \alpha]\rangle)$ in two polarizations
- $$\Omega_{S^1} = \sum_{\text{arcs } a} \Omega_a + \sum_{\text{vertices } v} \Omega_v + \sum_{(v,a)} \Omega_{v,a}$$

  Example: $\Omega(\ \overset{\mathbb{A}\quad\ \beta}{\longrightarrow\!\!\bullet_p}\ ) = -\sum_{j=0}^{\infty} \frac{B_j}{j!}\langle\beta, \left(i\hbar\frac{\partial}{\partial\beta}\right)^j \mathbb{A}_p\rangle$

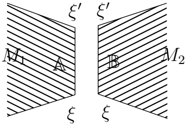
2d Yang-Mills: quantization with corners cont'd

One has Baez-Dolan-Lurie extended TQFT picture:

- points $\to$ dg algebras
- arcs $\to$ dg bimodules
- 2-cells $\to$ (area-dependent) morphisms of dg bimodules



One has:

- mQME for surfaces with corners
- Gluing of arcs at vertices $\quad \to$ (arc bimod)$_1 \otimes_{\mathcal{H}^{\xi}_{pt}}$ (arc bimod)$_2$
- Gluing of regions over arcs 

$\to$

$$\int \mathcal{D}\mathbb{A}_I \mathcal{D}\mathbb{B}_I \; Z_1(\cdots, \mathbb{A}_I) e^{-\frac{i}{\hbar}\int_I \langle \mathbb{B}_I, \mathbb{A}_I \rangle} Z_2(\mathbb{B}_I, \cdots)$$

Chern-Simons theory on a cylinder

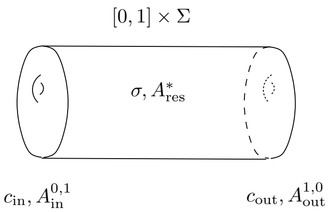
Reference: A. S. Cattaneo, P. M, K. Wernli, "Quantum Chern-Simons theories on cylinders: BV-BFV partition functions," arXiv:2012.13983

Chern-Simons theory on $M = I \times \Sigma$:

- $S = \int_M \mathrm{tr}\left(\frac{1}{2}\mathcal{A} \wedge d\mathcal{A} + \frac{1}{6}\mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]\right)$
- $\mathcal{A} = c + A + A^* + c^* \qquad \in \mathcal{F} = \Omega^\bullet(M, \mathfrak{g})[1]$

  $= \Omega^\bullet(I, \Omega^\bullet(\Sigma, \mathfrak{g}))[1]$

Chern-Simons theory on a cylinder, cont'd



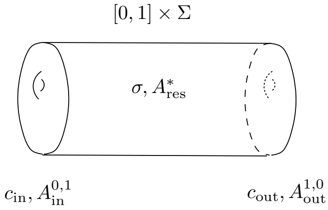
- Polarizations: $c, A^{0,1}$ fixed at $t=0$, $\quad c, A^{1,0}$ fixed at $t=1$.
  (Using a complex structure on Σ)
- Fiber of $\mathcal{F} \to \mathcal{B}$:

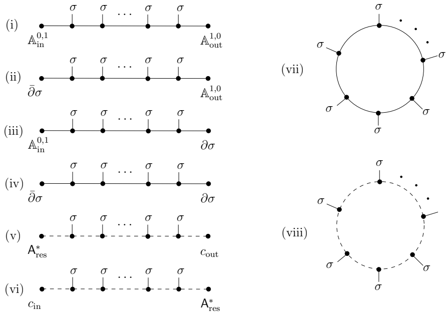
$$\mathcal{Y} = \Omega^\bullet(I, \partial I; \Omega^0_\Sigma) \oplus \Omega^\bullet(I, \{0\}; \Omega^{0,1}_\Sigma) \oplus \Omega^\bullet(I, \{1\}; \Omega^{1,0}_\Sigma) \oplus \Omega^\bullet(I; \Omega^2_\Sigma)$$

$$\Big\downarrow \text{def retract}$$

$$\mathcal{V} = H^\bullet(I, \partial I; \Omega^0_\Sigma) \oplus H^\bullet(I; \Omega^2_\Sigma) \quad = \{(dt \cdot \sigma, A^*_{\mathrm{res}})\}$$

CS on a cylinder: effective action





$$S^{\mathrm{eff}}=\int_\Sigma \langle A^{1,0}_{\mathrm{out}}, e^{-\mathrm{ad}_\sigma}\circ A^{0,1}_{\mathrm{in}}\rangle+\langle A^{1,0}_{\mathrm{out}},\frac{1-e^{-\mathrm{ad}_\sigma}}{\mathrm{ad}_\sigma}\circ\bar\partial\sigma\rangle+\langle A^{0,1}_{\mathrm{in}},\frac{e^{\mathrm{ad}_\sigma}-1}{\mathrm{ad}_\sigma}\circ\partial\sigma\rangle$$

$$-\langle\partial\sigma,\frac{e^{-\mathrm{ad}_\sigma}+\mathrm{ad}_\sigma-1}{(\mathrm{ad}_\sigma)^2}\circ\bar\partial\sigma\rangle-\langle A^*_{\mathrm{res}},\frac{\mathrm{ad}_\sigma}{1-e^{-\mathrm{ad}_\sigma}}\circ c_{\mathrm{out}}-\frac{\mathrm{ad}_\sigma}{e^{\mathrm{ad}_\sigma}-1}\circ c_{\mathrm{in}}\rangle$$

$$-\,i\hbar\,\mathrm{tr}_{C^\infty(\Sigma,\mathfrak{g})}\log\frac{\sinh\frac{\mathrm{ad}_\sigma}{2}}{\frac{\mathrm{ad}_\sigma}{2}}$$

CS on a cylinder: effective action as G/G WZW

Introduce the group parametrization for $\mathcal{V}$:

$$(\sigma, A^*_{\text{res}}) \quad \to \quad \Big(g = e^{-\sigma}, \ \ \underbrace{g^* = -\big(\tfrac{\text{ad}_{\log g}}{\text{Ad}_g - 1} \circ A^*_{\text{res}}\big) \cdot g^{-1}}_{\text{Darboux conjugate of } g}\Big)$$

In this parametrization:

$$S^{\text{eff}} = S_{\text{WZW}}(g) + \int_\Sigma \langle A^{1,0}_{\text{out}}, g A^{0,1}_{\text{in}} g^{-1}\rangle - \langle A^{1,0}_{\text{out}}, \bar\partial g \cdot g^{-1}\rangle - \langle A^{0,1}_{\text{in}}, g^{-1}\partial g\rangle$$
$$+ \int_\Sigma -\langle c_{\text{out}}, g g^*\rangle + \langle c_{\text{in}}, g^* g\rangle$$
$$= S_{G/G\ \text{WZW}} + \text{ghost term}$$

Here

$$S_{\text{WZW}}(g) = -\frac{1}{2}\int_\Sigma \langle \partial g \cdot g^{-1}, \bar\partial g \cdot g^{-1}\rangle - \frac{1}{12}\int_{I\times\Sigma} \langle d\tilde g \cdot \tilde g^{-1}, [d\tilde g \cdot \tilde g^{-1}, d\tilde g \cdot \tilde g^{-1}]\rangle$$

– Wess-Zumino-Witten action

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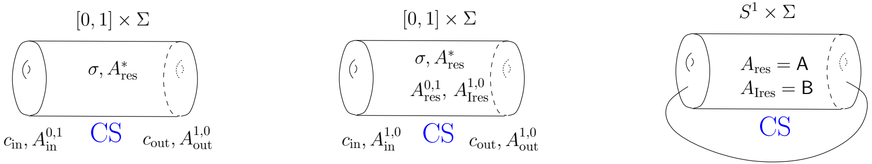
- mQME: $\boxed{(\Omega_{\text{out}}+\Omega_{\text{in}}+\hbar^2\Delta_{\mathcal{V}})e^{\frac{i}{\hbar}S^{\text{eff}}}=0}$
  with $\Omega_{\text{out}}=$
  $$\int_\Sigma \left\langle c_{\text{out}},\bar\partial A^{1,0}_{\text{out}}-i\hbar\left(\partial+[A^{1,0}_{\text{out}},-]\right)\frac{\delta}{\delta A^{1,0}_{\text{out}}}\right\rangle - i\hbar\left\langle \frac12[c_{\text{out}},c_{\text{out}}],\frac{\delta}{\delta c_{\text{out}}}\right\rangle,$$
  $\Delta_{\mathcal{V}}=\int_\Sigma \frac{\delta}{\delta\sigma}\frac{\delta}{\delta A^*_{\text{res}}}$
- mQME $\Leftrightarrow$ Polyakov-Wiegmann formula for $\mathbb{I}=S_{G/G\ \mathrm{WZW}}$:
  $$\mathbb{I}({}^{h_{\text{in}}}(A^{0,1}_{\text{in}}),{}^{h_{\text{out}}}(A^{1,0}_{\text{out}});h_{\text{out}}gh_{\text{in}}^{-1}) = \mathbb{I}(A^{0,1}_{\text{in}},A^{1,0}_{\text{out}};g)-\mathbb{I}(A^{0,1}_{\text{in}},0;h_{\text{in}})-\mathbb{I}(0,A^{1,0}_{\text{out}};h_{\text{out}}^{-1})$$
- $\mathbb{I}$ is a "generalized generating function" for the evolution Lagrangian
  $$L=\{(A_{\text{in}},A_{\text{out}})\mid \exists \text{ flat } A\in \mathrm{Conn}_{I\times\Sigma} \text{ s.t. } A|_{t=0}=A_{\text{in}},\ \ A|_{t=1}=A_{\text{out}}\}\quad\subset \overline{\mathrm{Conn}(\Sigma)}\times\mathrm{Conn}(\Sigma)$$
  I.e. $L=\{(A^{0,1}_{\text{in}},A^{1,0}_{\text{in}}=-\frac{\delta\mathbb{I}}{\delta A^{0,1}_{\text{in}}},A^{1,0}_{\text{out}},A^{0,1}_{\text{out}}=\frac{\delta\mathbb{I}}{\delta A^{1,0}_{\text{out}}})\mid \frac{\delta\mathbb{I}}{\delta g}=0\}$

G/G WZW, $\widetilde{\mathrm{WZW}}$ and 2d BF



↓ ↓ Kaluza–Klein reduction ↓

G/G WZW$_{+\mathrm{ghosts}}$ ⟵ $\widetilde{\mathrm{WZW}}$ $\xrightarrow{\text{gluing}}$ "compact" 2d BF

↓ pairing with boundary states

WZW

$$S_{\widetilde{\mathrm{WZW}}} = -S_{\mathrm{WZW}}(g^{-1}) + \int_\Sigma -\langle \Lambda, \partial g\cdot g^{-1}\rangle + \langle A^{1,0}_{\mathrm{out}}, \Lambda\rangle + \langle A^{1,0}_{\mathrm{in}}, g^{-1}\Lambda g + g^{-1}\bar\partial g\rangle$$
$$+ \langle c_{\mathrm{out}}, -gg^* + \bar\partial\Lambda^* + [\Lambda,\Lambda^*]\rangle + \langle c_{\mathrm{in}}, g^* g\rangle$$

$$(\sigma, A^*_{\mathrm{res}}; A^{0,1}_{\mathrm{res}}, A^{1,0}_{\mathrm{Ires}}) \xrightarrow{\text{symplectomorphism}} (g, g^*; \Lambda, \Lambda^*)$$

7d abelian CS – BCOV correspondence

Another example of effective CS on a cylinder.
7d abelian Chern-Simons $S = \frac{1}{2}\int \mathcal{A}\wedge d\mathcal{A}$, $\mathcal{A}\in\Omega^\bullet(M)[3]$,
$M = [0,1]\times X$, X Calabi-Yau, $\dim_{\mathbb{C}} X = 3$.

Pick linear chiral polarization at $t=0$ and
non-linear Hitchin polarization of $\Omega^3(X)$ at $t=1$

$$\langle\Psi_{\mathrm{out}}|Z|\Psi_{\mathrm{in}}\rangle = \int_{\Omega^{1,1}(X)} \mathcal{D}b\;\; e^{\frac{i}{\hbar}\int_X \frac{1}{2}\partial b\,\bar{\partial} b+\frac{1}{6}\langle\partial b,\partial b,\partial b\rangle}$$

R.h.s. $=$ BCOV path integral (a.k.a. Kodaira-Spencer gravity);
b – CS residual field

Reference: A. Gerasimov, S. Shatashvili, "Towards integrability of topological strings I: three-forms on Calabi-Yau manifolds," arXiv:hep-th/0409238;
A. S. Cattaneo, P. M., K. Wernli, "Quantum Chern-Simons theories on cylinders: BV-BFV partition functions," arXiv:2012.13983