Gone Fishing, ND
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Around AKSZ sigma models

Classical BV theory: $\mathcal{F}$ - space of fields ($\mathbb{Z}$-graded supermfd)

$S \in C^\infty(\mathcal{F})_0$ action

$\omega \in \Omega^2(\mathcal{F})_{-1}$ symplectic structure, $gh = -1$ (BV 2-form)

classical master equation: $\{S, S\}_\omega = 0$

induced object: $Q = \cancel{X_S} = \{S, -\}$ - cohom. vector field on $\mathcal{F}$, $gh = 1$

$Q^2 = 0$

AKSZ construction

fields: maps $\mathcal{M} \to \mathcal{N}$ i.e. $\mathcal{F} = Map(\mathcal{M}, \mathcal{N})$

dg manifolds with additional structure

source: $(\mathcal{M}, Q_{\mathcal{M}}, \mu)$

cohom. v.f.

Berezinian of degree $-n$

i.e. $\int_{\mathcal{M}} \mu(\ldots) : C^\infty(\mathcal{M})_n \longrightarrow \mathbb{R}$

$C^\infty(\mathcal{M})_{\neq n} \longrightarrow 0$

main example: $\mathcal{M} = T[1]M$

M - closed oriented n-mfd

$C^\infty(\mathcal{M}) \cong \Omega^\bullet(M)$

$Q_{\mathcal{M}} = d_M$ ← de Rham diff. on M

μ = integration of a form on M (top)

target: $(\mathcal{N}, \Theta, \omega_{\mathcal{N}} = \delta\alpha_{\mathcal{N}}, Q_{\mathcal{N}} = \{\Theta, \cdot\}_{\omega_{\mathcal{N}}})$

← a "degree $(n-1)$ Hamiltonian dg manifold"

Hamiltonian, $gh = n$

exact symp. form, $gh = n-1$

fixed primitive

satisfying $\{\Theta, \Theta\}_{\omega_{\mathcal{N}}} = 0$

$\boxed{\text{Ex: } \mathcal{N} \text{ - gr. v. space} \Rightarrow \mathcal{F} = \Omega^\bullet(M) \otimes \mathcal{N}, \quad \mathcal{M} = T[1]M}$

Construction using local coordinates x^a on $\mathcal{N}$, $u^i, \theta^i = du^i$ on $T[1]M$

$$X^a = \sum_{k=0}^{n} \sum_{0 \le i_1 < \ldots < i_k \le n} X^a_{i_1 \ldots i_k}(u)\, \theta^{i_1} \cdots \theta^{i_k}$$ - "superfield" corresp. to x^a

$\{X^a_{i_1 \ldots i_k}(u)\}$ - coordinates on $\mathcal{F}$, $gh\, X^a_{i_1 \ldots i_k} = gh(x^a) - k$

assuming $\omega_{\mathcal{N}} = \sum_{a,b} \omega_{ab}(x)\, \delta x^a \wedge \delta x^b = \delta\left(\sum_a \alpha_a(x)\, \delta x^a\right)$,

construct $$\omega = \int_M \sum_{a,b} \omega_{ab}(X)\, \delta X^a \wedge \delta X^b$$ ← BV 2-form on $\mathcal{F}$

← substitution of X^c instead of x^c in component functions

$$S = \int_M \underbrace{\sum_a \alpha_a(X)\, dX^a}_{\text{"kinetic"/source term}} + \underbrace{\Theta(X)}_{\text{interaction/target term}}$$ - AKSZ action

Coordinate-free construction:

$$\mathcal{M}\times \underbrace{\mathrm{Map}(\mathcal{M},\mathcal{N})}_{\mathcal{F}} \xrightarrow{ev} \mathcal{N}$$

$$p\downarrow \qquad \mathrm{Map}(\mathcal{M},\mathcal{N})$$

here $p_* = \int_{\mathcal{M}} \mu(\dots) : C^\infty(\mathcal{M})_n \to \mathbb{R}$

"Transgression"

$$\mathbb{T} = p_* ev^* : \Omega^q(\mathcal{N})_j \to \Omega^q(\mathcal{F})_{j-n}$$

$$\psi = \sum_{a_1\dots a_q} \psi_{a_1\dots a_q}(x)\delta x^{a_1}\dots\delta x^{a_q} \mapsto$$

$$\mapsto \mathbb{T}\psi = \int_{\mathcal{M}} \sum_{a_1\dots a_q} \psi_{a_1\dots a_q}(X)\delta X^{a_1}\dots\delta X^{a_q}\mu = \int_M (-\text{"}-)$$

Note: the superfields are $X^a = ev^*(x^a)$

- BV 2-form: $\omega = \mathbb{T}\underbrace{\omega_{\mathcal{N}}}_{\in\Omega^2(\mathcal{N})_{n-1}} \in \Omega^2(\mathcal{F})_{-1}$

~~action~~ $S_{\mathcal{F}}$

- lift $Q_{\mathcal{M}} = d_M$ and $Q_{\mathcal{N}}$ to coh. v. fields on $\mathrm{Map}(\mathcal{M},\mathcal{N})$

$Q^{total}_{\mathcal{F}} = Q^{lifted}_{\mathcal{M}} + Q^{lifted}_{\mathcal{N}}$ - coh. v.f. on $\mathcal{F}$ ("BRST operator")

- action:

$$S = \underbrace{\iota_{Q^{lifted}_{\mathcal{M}}}}_{\delta X^a \mapsto dX^a} \underbrace{\mathbb{T}\alpha_{\mathcal{N}}}_{\int_M \sum_a \alpha_a(X)\delta X^a} + \underbrace{\mathbb{T}\Theta}_{\int_M \Theta(X)} \in C^\infty(\mathcal{F})_0$$

Thm: $(\mathcal{F},\omega,S,Q)$ is a BV package, in particular;
$\{S,S\}=0$ - CME
$Q^2=0$
$L_Q \omega = \delta S$

- $\mathcal{F}\supset F_{cl} = \mathrm{Map}_0(\mathcal{M},\mathcal{N})$ - class. fields $\supset EL = \mathrm{Map}_{0,dg}(\mathcal{M},\mathcal{N})$ - solutions of Euler-Lagrange equations $\delta S=0$ $\iff$ zero-locus of Q

$\mathrm{Map}_1(\mathcal{M},\mathcal{N})$ - infinitesimal symmetries

- EL-moduli space $\mathcal{M} = \mathrm{zero}(Q)/Q\text{-distribution}$ $\qquad \mathcal{M}^{gh=0} = $ solutions of E-L eq. / gauge symmetry

$\quad \Vert$ $(\mathcal{EL},\underline{\omega})$ odd-symplectic

Ex: Chern-Simons theory $n=3$, $\mathcal{M} = T[1]\underset{\text{oriented closed 3-mfld}}{M}$

G - simply connected Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $\langle x,y\rangle = \mathrm{tr}\,xy$ - Killing form, $\{T_a\}$ - o/n basis in $\mathfrak{g}$, f^{abc} - structure constants

$\rightsquigarrow$ target: $\mathcal{N} = \mathfrak{g}[1]$, $\omega_{\mathcal{N}} = \sum_a \frac{1}{2}\delta\psi^a\wedge\delta\psi^a = \frac{1}{2}\langle\delta\psi,\delta\psi\rangle = \delta\alpha_{\mathcal{N}}$

$\sum \psi^a T^a = \psi$ $\qquad \alpha_{\mathcal{N}} = \sum_a \frac{1}{2}\psi^a\delta\psi^a = \frac{1}{2}\langle\psi,\delta\psi\rangle$

$\Theta = \frac{1}{6}\sum_{abc} f^{abc}\psi^a\psi^b\psi^c \in C^\infty(\mathfrak{g}[1])_3$ $\qquad Q_{\mathcal{N}} = \sum \frac{1}{2} f^{abc}\psi^a\psi^b\frac{\partial}{\partial\psi^c}$ - Chevalley-Eilenberg differential

$\qquad = \frac{1}{6}\langle\psi,[\psi,\psi]\rangle$ $\qquad = \langle\frac{1}{2}[\psi,\psi],\frac{\partial}{\partial\psi}\rangle$ on $C^\infty(\mathfrak{g}[1]) \cong \wedge^\bullet\mathfrak{g}^* = C^\bullet_{CE}(\mathfrak{g})$

Space of fields: $\mathcal{F} = \mathrm{Map}(T[1]M,\mathfrak{g}[1]) \cong \Omega^\bullet(M,\mathfrak{g})[1]$

superfield: $\mathcal{A} = \mathcal{A}^a T^a = \mathcal{A}_{(0)} + \mathcal{A}_{(1)} + \mathcal{A}_{(2)} + \mathcal{A}_{(3)}$ $\leftarrow$ degree as a form on M

$\qquad\qquad 1 \quad 0 \quad -1 \quad -2$ $\leftarrow$ ghost number

$$= \underbrace{c}_{\text{ghost}} + \underbrace{A}_{\text{class field}} + A^+ + c^+$$

infin. gauge transf. $\qquad$ Lie algebra of gauge symm.

$$\omega = \int_M \frac{1}{2}\langle\delta\mathcal{A},\delta\mathcal{A}\rangle = \int_M \mathrm{tr}(\delta A\wedge\delta A^+ + \delta c\wedge\delta c^+)$$

$$S = \int_M \mathrm{tr}\left(\frac{1}{2}\mathcal{A}\wedge d\mathcal{A} + \frac{1}{6}\mathcal{A}\wedge[\mathcal{A},\mathcal{A}]\right) = \underbrace{\int \mathrm{tr}\left(\frac{1}{2}A\wedge dA + \frac{1}{6}A\wedge[A,A]\right)}_{S_{cl}} + \int \mathrm{tr}\, A^+\wedge d_A c + \int \mathrm{tr}\,\frac{1}{2} c^+\wedge[c,c]$$

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$$Q = \int_M \langle dA + \tfrac{1}{2}[A,A], \tfrac{\delta}{\delta A} \rangle = \int_M \langle F_A, \tfrac{\delta}{\delta A^+} \rangle + \dots$$

<u>moduli space</u> $\mathcal{M}^{gh=0} = MFC(M,G) = \mathrm{Hom}(\pi_1(M), G)/G$

$\hat{\mathcal{M}}$ – "thickening", odd-symplectic, $\quad T_{[A]}\mathcal{M} = H^\bullet_{d_A}(\Omega(M,\mathfrak{g}))[1]$ ← class of a flat connection

Ex:

<u>Poisson sigma model:</u> $\mathcal{M} = T[1]\Sigma$ (← 2-dimensional), $\quad \mathcal{N} = T^*[1]N$ (x^i – loc. coord), $\quad (N,\pi)$ – Poisson mfld

target structure: $\alpha_{\mathcal{N}} = \sum_i p_i \delta x^i$

$\Theta_{\mathcal{N}} = \sum_{i,j} \tfrac{1}{2}\pi^{ij}(x) p_i p_j$

fields:

gh \ form degree	0	1	2
1	$P_{(0)} = \beta$		
0	$X = X_{(0)}$	$P_{(1)} = \xi$	
−1		$\xi^+ = X_{(1)}$	$P_{(2)} = X^+$
−2			$\beta^+ = X_{(2)}$

← $\tilde{P}$, $\tilde{X}$

$P_{(k)} \in \Gamma(\Sigma, \Lambda^k T^*\Sigma \otimes X^* T^*N)$

$X_{(k\geq 1)} \in \Gamma(\Sigma, \Lambda^k T^*\Sigma \otimes X^* TN)$

$(\tilde{X}, \tilde{P}) \in \mathcal{F} = \mathrm{Map}(T[1]\Sigma, T^*[1]N)$

$$F = \mathcal{F}^{gh=0} = \left\{ \begin{array}{ccc} T\Sigma & \xrightarrow{\xi} & T^*N \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{X} & N \end{array} \right\}$$

$$S = \int_{\mathcal{M}} \sum_i \tilde{P}_i \, d\tilde{X}^i + \sum_{i,j} \tfrac{1}{2}\pi^{ij}(\tilde{X}) \tilde{P}_i \tilde{P}_j$$

$EL = \mathrm{Hom}_{\text{LieAlgbd}}(T\Sigma, T^*N)$

- AKSZ on manifolds with boundary

  > "BV-BFV formalism"

  M – an n-mfld with bdry

~~Σ – closed $(n-1)$-dim $\rightsquigarrow (\Phi_\Sigma, \omega_\Sigma = \delta\alpha_\Sigma, Q_\Sigma, S$~~

$$\begin{array}{ccccccccc} M & \rightsquigarrow & (\mathcal{F}, & \underset{-1}{\omega}, & Q, & \underset{0}{S}) & \leftarrow \text{space of (BV) fields} \\ & & \downarrow \pi & & \downarrow \pi_* & & \\ \partial M & \rightsquigarrow & (\Phi_\partial, & \underset{0}{\omega_\partial} = \delta\alpha_\partial, & \underset{1}{Q_\partial}, & \underset{-1}{S_\partial}) & \leftarrow \text{(BFV) phase space} \end{array}$$

relations: $Q^2 = Q_\partial^2 = 0$, $\quad Q_\partial = \{S_\partial, -\}_{\omega_\partial}$,

$$\boxed{\iota_Q \omega = \delta S + \pi^* \alpha_\partial} \Longrightarrow \left. \begin{array}{l} \bullet\ Q(S) = \pi^*(2S_\partial - \iota_{Q_\partial}\alpha_\partial) \\ \tfrac{1}{2}\iota_Q \iota_Q \omega = \pi^* S_\partial \end{array} \right\} \text{"CME"}$$

- $L_Q \omega = \pi^* \omega_\partial$

BV-BFV theory

- can be cast as a functor

cobordisms $(\partial_{in}\ M\ \partial_{out}) \longrightarrow$

$$\begin{array}{ccc} & \mathcal{F} & \\ \swarrow & & \searrow \\ \Phi_{in} & & \Phi_{out} \end{array}$$

gluing → (homotopy) fiber products.

- Euler-Lagrange ~~reduction~~ spaces $\quad \mathcal{EL} \subset \mathcal{F}$

$$\begin{array}{ccccc} & \mathcal{EL} & \subset & \mathcal{F} \\ \swarrow & & & \downarrow \pi \\ \underset{\text{Lagrangian}}{\mathcal{L}} \subset & \underset{\text{zero } Q_\partial}{\mathcal{EL}_\partial} & \subset & \Phi_\partial \end{array}$$

coisotropic, symplectic

reduction: (moduli spaces)

$$\begin{array}{l} \mathcal{M}, \Pi \quad (\Pi: gh=+1 \text{ Poisson}) \\ \downarrow \pi_* \\ \mathcal{M}_\partial, \underset{\text{symplectic}}{\omega_\partial} \end{array}$$

- $\mathcal{L} = \mathrm{im}\, \pi_* \subset \mathcal{M}_\partial$ Lagrangian
- Symp. leaves of Π = fibers of π_*

for an AKSZ theory: (M with bdry):

$\mathcal{F} = \mathrm{Map}(T[1]M, \mathcal{N})$ $\qquad \omega, Q, S$ as before

$\Phi_\partial = \mathrm{Map}(T[1]\partial M, \mathcal{N})$ $\qquad \alpha_\partial = \mathbb{T}_\partial \alpha_{\mathcal{N}}$, $\quad S = \int_{\partial M} \sum_a \alpha_a(X_\partial) X_\partial^a + \Theta(X_\partial)$

$\pi: X \mapsto X_\partial$ – pullback of forms to the bdry

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ex: gh=0 part for Chern-Simons on a 3-mfld M with bdry $\partial M = \Sigma$

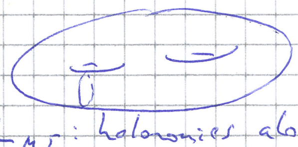
$$\Phi_\Sigma = \Omega^\bullet(\Sigma, \mathfrak{g})[1] \qquad \mathcal{A}_\partial = \mathcal{A}_\partial^{(0)} + \mathcal{A}_\partial^{(1)} + \mathcal{A}_\partial^{(2)} = c_\partial + A_\partial + A_\partial^+$$

$$\alpha_\partial = \int_\Sigma \tfrac{1}{2} \mathrm{tr}\, \mathcal{A} \wedge \delta \mathcal{A} \qquad S_\partial = \int_\Sigma \mathrm{tr}\, \tfrac{1}{2} \mathcal{A} \wedge d_\Sigma \mathcal{A} + \tfrac{1}{6} \mathcal{A} \wedge [\mathcal{A}, \mathcal{A}] = \int_\Sigma \mathrm{tr}\, c\, F_A + \tfrac{1}{2} A^+ [c, c]$$

$$= \int_\Sigma \mathrm{tr}\, \tfrac{1}{2} A \delta A + \tfrac{1}{2} c \delta A^+ + \tfrac{1}{2} A^+ \delta c$$

gh=0 part of EL spaces:

$$\begin{array}{ccccc} & & \underbrace{\mathrm{FlatConn}_M}_{EL_M} & \subset & \mathrm{Conn}_M \\ & \swarrow^{\pi} & & & \downarrow \pi \\ \underbrace{L_{M,\Sigma}}_{\text{conn. on } \Sigma \text{ extendable as flat into } M.} & \subset & \underbrace{\mathrm{FlatConn}_\Sigma}_{EL_\Sigma} & \subset & \mathrm{Conn}_\Sigma \end{array} \quad \xrightarrow{\text{reduction}} \quad \begin{array}{c} \mathcal{M}_M^{gh=0} = MFC(M, G) \\ \downarrow \pi_* \\ \underbrace{L_{M,\Sigma}}_{\text{Lagrangian}} \subset \mathcal{M}_\Sigma^{gh=0} = MFC(\Sigma, G) \end{array}$$



$L_{M,\Sigma}$: holonomies along loops contractible in M are trivial

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Perturbative answer in Chern-Simons theory, M closed
(Witten - Axelrod - Singer)

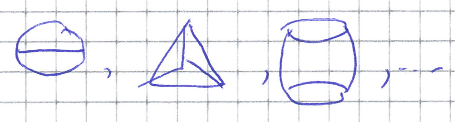
$$\left[\int_{\mathrm{Conn}_M} DA\; e^{\frac{i}{\hbar} S_{CS}(A)}\right]_{\text{pert., near } A_0 \text{ - acyclic flat conn.}} \xrightarrow{\text{gauge-fixing}} \int_{\substack{\mathcal{L} \subset \mathcal{F} \\ d_{A_0}^* a = 0}} D\tilde{a}\; e^{\frac{i}{\hbar} S_{CS}(A + \tilde{a})} \quad \xrightarrow{\text{perturbative evaluation}}$$

$A = A_0 + a$

$$\rightsquigarrow Z^{pert}(M, G, A_0, \varphi, \hbar) = e^{\frac{i}{\hbar} S_{CS}(A_0)} \cdot \left(\tau(M, A_0)^{\frac{1}{2}} e^{\frac{\pi i}{4} \eta(\ldots)}\right) \cdot Z_{\geq 2 \text{ loops}} \cdot e^{\ldots\, c(\hbar) S_{grav}(g, \varphi)}$$

φ: framing of M; τ: R-torsion; η: APS η-invariant of $L_- = d_{A_0} * + * d_{A_0}$ on $\Omega^{odd}(M, \mathfrak{g})$

$$Z_{\geq 2 \text{ loops}} = \exp \sum_{\substack{\text{3-valent} \\ \text{connected graphs } \Gamma}} \frac{(i\hbar)^{-\chi(\Gamma)}}{|\mathrm{Aut}\,\Gamma|} \Phi_\Gamma$$

Γ: 

configuration space integral

$$\int_{\mathrm{Conf}_V(M)} \langle \underbrace{\prod_{\text{edges } (uv)} \pi_{uv}^* \eta}_{E}, \underbrace{f \otimes \cdots \otimes f}_{V} \rangle$$

structure constants of $\mathfrak{g}$

propagator
$\eta \in \Omega^2(\mathrm{Conf}_2(M), \mathfrak{g} \otimes \mathfrak{g})$
- integral kernel for operator $d_{A_0}^* \Delta_{A_0}^{-1} : \Omega(M) \circlearrowleft$

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quantum BV-BFV theory (on M with bdry)

$\partial M \rightsquigarrow (\mathcal{H}_\partial^\bullet, \Omega_\partial)$ ← q. BFV charge - differential

$M \rightsquigarrow$
- $\mathcal{F}_{res}, \omega_{res}$
- $Z \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{res}) \otimes \mathcal{H}_\partial^\bullet$

mQME: $\left(i\hbar \Delta_{res} + \frac{i}{\hbar} \Omega_\partial\right) Z = 0$

idea of quantization:

$$\begin{array}{l} \mathcal{F} \\ \downarrow \pi \\ \Phi_\partial \\ \downarrow p \quad \text{- Lagr. fibration compatible with } \alpha_\partial \\ \mathcal{B}_\partial \end{array} \qquad \rightsquigarrow \mathcal{H}_\partial = \mathrm{Dens}^{\frac{1}{2}} \mathcal{B}_\partial, \quad \Omega_\partial = \hat{S}_\partial$$

$$\mathcal{F}_b = \pi^{-1} p^{-1} \{b\} \simeq \mathcal{F}_{res} \times \mathcal{V}, \quad b \in \mathcal{B}_\partial$$

$$Z(b, \varphi_{res}) = \int_{\mathcal{L} \subset \mathcal{V}} D\varphi_{fluct}\; e^{\frac{i}{\hbar} S(b + \varphi_{res} + \varphi_{fluct})}$$

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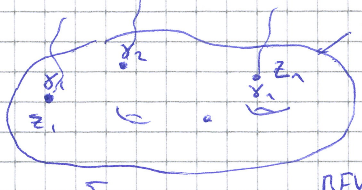
Boundary structure in quantum Chern-Simons, $\Sigma = \partial M$



Σ

punctures decorated with γ_i - irreps of G

coad. orbits $\subset \mathfrak{g}^*$

$$\Phi_\Sigma = (\underbrace{\Omega^0 \oplus \Omega^1}_{\Omega^0 \oplus \Omega^{0,1} =: \mathcal{B}} \oplus \Omega^2) \times \prod_i \mathcal{O}_{\gamma_i}$$

- need a complex structure on Σ.

$\mathcal{H}_\Sigma^{BFV} \ni \psi(c, A^{0,1})$ with values in $R_{\gamma_1} \otimes \cdots \otimes R_{\gamma_n}$ - representation space

$$\Omega_\Sigma^{BFV} = \int_\Sigma \left\langle c, \partial A^{0,1} + \hbar \bar\partial \frac{\delta}{\delta A^{0,1}} + \hbar \left[A^{0,1}, \frac{\delta}{\delta A^{0,1}} \right] \right\rangle + \hbar \left\langle \tfrac{1}{2}[c,c], \frac{\delta}{\delta c} \right\rangle \; ; \; \hbar = \frac{\pi}{k}$$

$$+ \sum_i \underbrace{\rho_i(c(z_i))}_{c^a(z_i)\rho_i(T^a)}$$

Gawędzki-Kupiainen/
Alekseev-Barmaz-M

Axelrod-Della Pietra-Witten

Thm: $H^0_{\Omega_\Sigma}(\mathcal{H}_\Sigma) =$ Verlinde space$_{0,n}$ $=$ GeomQ$(MFC_{\Sigma,G})$
(for genus $= 0$)
$G = SU(2)$

$=$ space of WZW conformal blocks on $\Sigma_{0,n}$

$= \mathrm{Hom}(\hat{R}_{\gamma_1} \otimes \cdots \otimes \hat{R}_{\gamma_n}, \mathbb{C})^{\mathfrak{g}(z_1 \cdots z_n)}$

$\leftarrow$ $\mathfrak{g}$-valued merom. functions on Σ with poles allowed at $z_1 \cdots z_n$

integrable irrep. of $\hat{\mathfrak{g}}_k$

Meta-conjecture: ~~for~~ in [good cases], we have a "quantization commutes with reduction" diagram

$$\begin{tikzcd}
(\Phi_\Sigma^{BFV}, Q_\Sigma^{BFV}, S_\Sigma^{BFV}) \arrow[r, "\text{canonical}", "\text{quantization}"'] \arrow[d, "\text{moduli space of a dg mfd}"'] & \mathcal{H}_\Sigma^{BFU}, \Omega_\Sigma^{BFV} \arrow[d, "H^0_{\Omega_\Sigma}"] \\
(M_\Sigma, \omega_\Sigma) \arrow[r, "\text{geom.}", "\text{quant.}"'] & \mathcal{H}_\Sigma^{red}
\end{tikzcd}$$

$\leftarrow$ small model for the space of states

How does it work for PSM, Σ - a circle or an interval?

> AKSZ $\to$ CPTVV

dg manifolds $\to$ derived stacks
- e.g. helps? to deal with non-simply connected G in Chern-Simons