Perturbative topological field theory with Segal-like gluing

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Joint work with Alberto S. Cattaneo and Nikolai Reshetikhin

Plan

1. Introduction: calculating partition functions by cut/paste.

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2. BV-BFV formalism for gauge theories on manifolds with boundary: an outline.

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3. Abelian BF theory in BV-BFV formalism.

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2. BV-BFV formalism for gauge theories on manifolds with boundary: an outline.
3. Abelian BF theory in BV-BFV formalism.
4. Further examples: Poisson sigma model, cellular models.

Introduction: calculating partition functions by cut/paste.

Idea:

$Z\Big($  $\Big) = \Big\langle Z\Big($  $\Big) , Z\Big($  $\Big)\Big\rangle$

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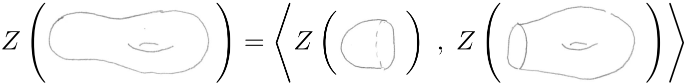
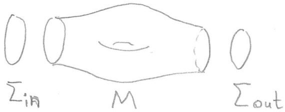
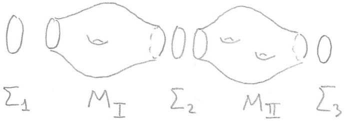


Functorial description (Atiyah-Segal):

Closed $(n-1)$-manifold Σ	$\mathcal{H}_\Sigma$
n-cobordism M 	Partition function $Z_M : \mathcal{H}_{\Sigma_{\text{in}}} \to \mathcal{H}_{\Sigma_{\text{out}}}$
Gluing 	Composition $Z_{M_I \cup M_{II}} = Z_{M_{II}} \circ Z_{M_I}$

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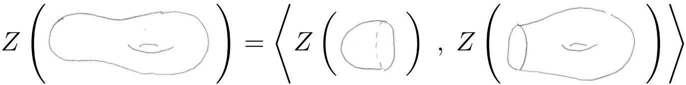

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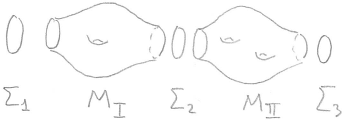
Gluing 	Composition $Z_{M_I \cup M_{II}} = Z_{M_{II}} \circ Z_{M_I}$

Atiyah: TQFT is a functor of monoidal categories $(\text{Cob}_n, \sqcup) \to (\text{Vect}_\mathbb{C}, \otimes)$.

Example: 2D TQFT

$Z\Big($  $\Big)$

can be expressed in terms of building blocks:

1. $Z\Big($  $\Big)\ :\ \mathbb{C} \to \mathcal{H}_{S^1}$
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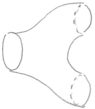
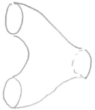
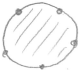
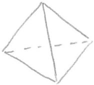
3. $Z\Big($  $\Big)\ :\ \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1}$
4. $Z\Big($  $\Big)\ :\ \mathcal{H}_{S^1} \to \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1}$

– Universal local building blocks for 2D TQFT!

For $n > 2$ we want to glue along pieces of boundary/ glue-cut with corners.

Building blocks: balls with stratified boundary (cells)





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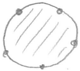
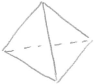
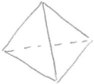




Extension of Atiyah's axioms to gluing with corners: extended TQFT (Baez-Dolan-Lurie).
Example: Turaev-Viro 3D state-sum model.

building block - 3-simplex 	q6j-symbol
gluing	sum over spins on edges

Problems:

- Very few examples!
- Some natural examples do not fit into Atiyah axiomatics.

Goal:

- Construct TQFT with corners and gluing out of perturbative path integrals for diffeomorphism-invariant action functionals.
- Investigate interesting examples.

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Note: $\{S, S\}_\omega = 0$.

BV-BFV formalism for gauge theories on manifolds with boundary

Reference: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332 2 (2014) 535–603.

For M with boundary:

$$\begin{array}{ccccc} M & \longrightarrow & (\mathcal{F}, \quad \omega, & Q, \; S) & \text{– space of fields} \\ & & \downarrow \pi \qquad\qquad & \downarrow \pi_* & \\ \partial M & \longrightarrow & (\mathcal{F}_\partial, \omega_\partial = \delta\alpha_\partial, & Q_\partial, S_\partial) & \text{– phase space} \end{array}$$

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Subscripts = "ghost numbers".

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Relations: $Q_\partial^2 = 0,\ \iota_{Q_\partial}\omega_\partial = \delta S_\partial;\qquad Q^2 = 0,\ \boxed{\iota_Q \omega = \delta S + \pi^* \alpha_\partial}.$

$\Rightarrow$CME: $\frac{1}{2}\iota_Q \iota_Q \omega = \pi^* S_\partial$

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This picture extends to higher-codimension strata!

Example: abelian Chern-Simons theory, $\dim M = 3$.

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Superfield $\mathcal{A} = \underbrace{c}_{\text{ghost},1} + \underbrace{A}_{\text{classical field},0} + \underbrace{A^+_{-1} + c^+_{-2}}_{\text{antifields}}$

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Euler-Lagrange moduli spaces:

$$\begin{array}{ccc} M & \longrightarrow & H^\bullet(M)[1] \\ & & \iota^* \Big\downarrow \\ \partial M & \longrightarrow & H^\bullet(\partial M)[1] \end{array}$$

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$*_\Sigma$ — pairing of states in $\mathcal{H}_\Sigma$,
P_* — BV pushforward (fiber BV integral) for
$\mathcal{F}^{M_I}_{\mathrm{res}} \times \mathcal{F}^{M_{II}}_{\mathrm{res}} \xrightarrow{P} \mathcal{F}^{M_I \cup_\Sigma M_{II}}_{\mathrm{res}}$

Quantization

Choose $p : \mathcal{F}_\partial \to \mathcal{B}$ Lagrangian fibration, $\alpha_\partial|_{p^{-1}(b)} = 0$.

$$\boxed{\mathcal{H}_\partial = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B})}, \ \Omega_\partial = \widehat{S_\partial} \quad \in \mathrm{End}(\mathcal{H}_\partial)_1.$$

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Partition function:

$$Z_M(b) = \int_{\mathcal{L}\subset\mathcal{F}_b} e^{\frac{i}{\hbar}S}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B})$$

$\mathcal{L} \subset \mathcal{F}_b$ gauge-fixing Lagrangian.
Problem: Z_M may be ill-defined due to zero-modes.

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Solution: Split $\mathcal{F}_b = \mathcal{F}_{\mathrm{res}} \times \widetilde{\mathcal{F}} \ni (\phi_{\mathrm{res}}, \widetilde{\phi})$. Partition function:

$$Z_M(b, \phi_{\mathrm{res}}) = \int_{\mathcal{L} \subset \widetilde{\mathcal{F}}} e^{\frac{i}{\hbar} S(b, \phi_{\mathrm{res}}, \widetilde{\phi})}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}) \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{\mathrm{res}})$$

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$$\begin{array}{ccl} \mathcal{F} \supset \mathcal{F}_b & = \pi^{-1} p^{-1}\{b\} \\ \pi \downarrow & \\ \mathcal{F}_\partial & \\ p \downarrow & \\ \mathcal{B} & \ni b \text{ boundary condition} \end{array}$$

Solution: Split $\mathcal{F}_b = \mathcal{F}_{\mathrm{res}} \times \widetilde{\mathcal{F}} \ni (\phi_{\mathrm{res}}, \widetilde{\phi})$. Partition function:

$$Z_M(b, \phi_{\mathrm{res}}) = \int_{\mathcal{L} \subset \widetilde{\mathcal{F}}} e^{\frac{i}{\hbar} S(b, \phi_{\mathrm{res}}, \widetilde{\phi})}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}) \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{\mathrm{res}})$$

$\mathcal{L} \subset \widetilde{\mathcal{F}}$ gauge-fixing Lagrangian.

$$\mathcal{F}_{\mathrm{res}} \xrightarrow{P} \mathcal{F}'_{\mathrm{res}} \qquad \Rightarrow \qquad Z'_M = P_* Z_M$$

Abelian BF theory: the continuum model.

Input:

- M a closed oriented n-manifold M.
- E an $SL(m)$-local system.

Abelian BF theory: the continuum model.
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Space of BV fields: $\mathcal{F} = \Omega^\bullet(M, E)[1] \oplus \Omega^\bullet(M, E^*)[n-2] \quad \ni (A, B)$
Action: $S = \int_M \langle B, d_E A \rangle$.

Reference: A. S. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants,* Lett. Math. Phys. 2, 3 (1978) 247–252.

A. S. Schwarz: For M closed and E acyclic, the partition function is the R-torsion $\tau(M, E) \in \mathbb{R}$.

Result, C-M-R arXiv:1507.01221

For M closed, E possibly non-acyclic,
$\mathcal{F}_{\text{res}} = H^\bullet(M,E)[1] \oplus H^\bullet(M,E^*)[n-2]$ and

$$Z_M = \xi \cdot \tau(M,E)$$

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$$\xi = (2\pi\hbar)^{\sum_{k=0}^{n}(-\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot\dim H^k(M,E)} \cdot (e^{-\frac{\pi i}{2}}\hbar)^{\sum_{k=0}^{n}(\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot\dim H^k(M,E)}$$

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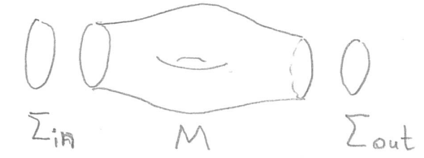
In particular Z_M contains a $\mathrm{mod} 16$ phase $e^{\frac{2\pi i}{16}s}$ with
$s = \sum_{k=0}^{n}(-1+2k(-1)^k)\cdot \dim H^k(M,E)$.

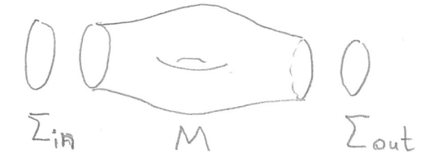
Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,



Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,



$$Z_M = \xi \cdot \tau(M, \Sigma_{\text{in}}; E) \cdot$$

$$\cdot \exp \frac{i}{\hbar} \left(\int_{\Sigma_{\text{out}}} \mathbb{B}\mathsf{a} + \int_{\Sigma_{\text{in}}} \mathsf{b}\mathbb{A} - \int_{\Sigma_{\text{out}} \times \Sigma_{\text{in}} \ni (x,y)} \mathbb{B}(x)\eta(x,y)\mathbb{A}(y) \right)$$

Result, C-M-R arXiv:1507.01221

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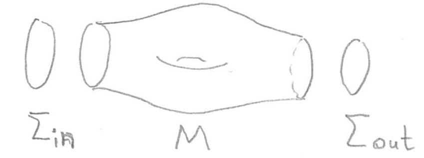

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Where: $\mathcal{F}_{\text{res}} = H^\bullet(M, \Sigma_{\text{in}}; E)[1] \oplus H^\bullet(M, \Sigma_{\text{out}}; E^*)[n-2] \ni (\mathsf{a}, \mathsf{b})$

Result, C-M-R arXiv:1507.01221

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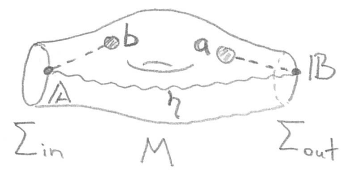

$$Z_M = \xi \cdot \tau(M, \Sigma_{\text{in}}; E) \cdot$$
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Where: $\mathcal{B} = \Omega^\bullet(\Sigma_{\text{in}})[1] \oplus \Omega^\bullet(\Sigma_{\text{out}})[n-2] \ni (\mathbb{A}, \mathbb{B})$

Result, C-M-R arXiv:1507.01221

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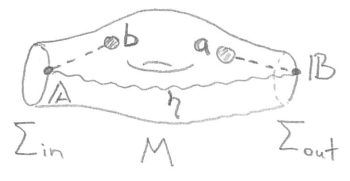
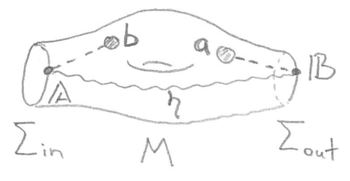


$$Z_M = \xi \cdot \tau(M, \Sigma_{\text{in}}; E) \cdot$$
$$\cdot \exp \frac{i}{\hbar} \left(\int_{\Sigma_{\text{out}}} \mathbb{B}\mathsf{a} + \int_{\Sigma_{\text{in}}} \mathsf{b}\mathbb{A} - \int_{\Sigma_{\text{out}} \times \Sigma_{\text{in}} \ni (x,y)} \mathbb{B}(x)\eta(x,y)\mathbb{A}(y) \right)$$

Where: ξ as before (but for relative cohomology),

Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,



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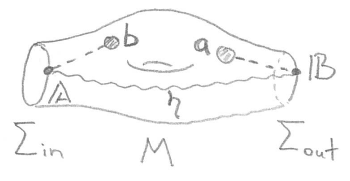
Where: τ - relative R-torsion,

Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,



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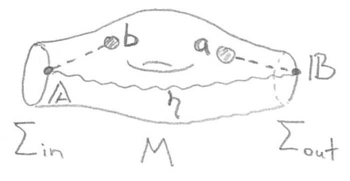
Where: $\eta \in \Omega^{n-1}(\mathrm{Conf}_2(M), E \boxtimes E^*)$ – propagator.

Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,



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This result satisfies:

- gluing

Result, C-M-R arXiv:1507.01221

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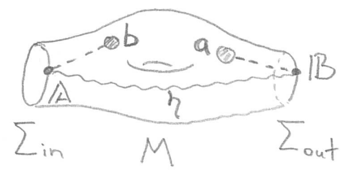

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This result satisfies:

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- mQME

Result, C-M-R arXiv:1507.01221

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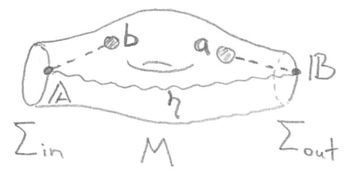

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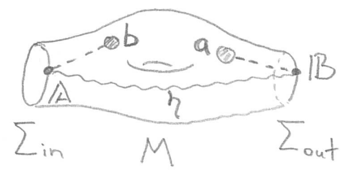
This result satisfies:

- gluing
- mQME
- change of η shifts Z_M by $\left(\frac{i}{\hbar}\Omega_\partial - i\hbar\Delta_{\text{res}}\right)$-exact term.

Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,



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This result satisfies:

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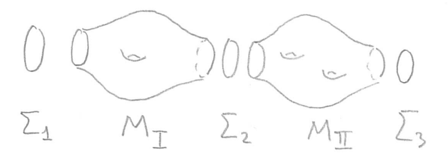
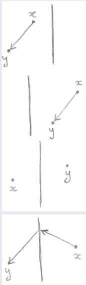
BFV operator: $\Omega_\partial = -i\hbar \left(\int_{\Sigma_{\text{out}}} d_E \mathbb{B} \frac{\delta}{\delta \mathbb{B}} + \int_{\Sigma_{\text{in}}} d_E \mathbb{A} \frac{\delta}{\delta \mathbb{A}} \right)$

Result, C-M-R arXiv:1507.01221



η_I, η_{II} – propagators on M_I, M_{II}.
Assume $H^\bullet(M,\Sigma_1) = H^\bullet(M_I,\Sigma_1) \oplus H^\bullet(M_{II},\Sigma_2)$.
Then the glued propagator on M is:

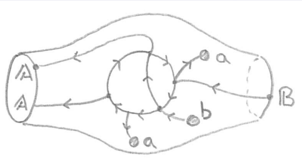
$$\eta(x,y) = \begin{cases} \eta_I(x,y) & \text{if } x,y \in M_I \\ \eta_{II}(x,y) & \text{if } x,y \in M_{II} \\ 0 & \text{if } x \in M_I, y \in M_{II} \\ \boxed{\int_{z\in\Sigma_2} \eta_{II}(x,z)\eta_I(z,y)} & \text{if } x \in M_{II}, y \in M_I \end{cases}$$



Example: Poisson sigma model, $n = 2$.
Action: $S = \int_M \langle B, dA \rangle + \frac{1}{2} \langle \pi(B), A \otimes A \rangle$
$\pi = \sum_{ij} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ Poisson bivector on $\mathbb{R}^m$.

Result, C-M-R arXiv:1507.01221

$$Z_M = \xi \cdot \tau \cdot \exp \frac{i}{\hbar} \sum_{\text{graphs}}$$



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Result, C-M-R arXiv:1507.01221

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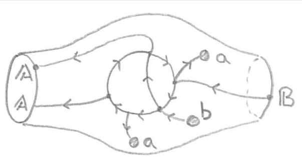

Z_M satisfies:

- gluing
- mQME
- change of η shifts Z_M by $\left(\frac{i}{\hbar}\Omega_\partial - i\hbar\Delta_{\text{res}}\right)$-exact term.

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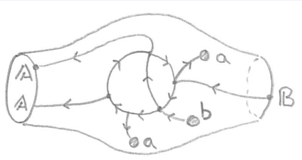

Z_M satisfies:

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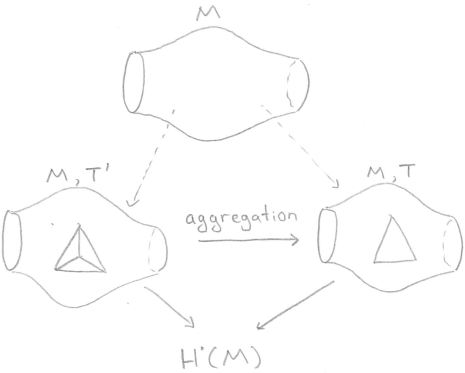
Ω_∂ = standard-ordering quantization ($\mathbb{B} \mapsto -i\hbar\frac{\delta}{\delta\mathbb{A}}$ on Σ_{in}, $\mathbb{A} \mapsto -i\hbar\frac{\delta}{\delta\mathbb{B}}$ on Σ_{out}) of $\boxed{\int_\partial \mathbb{B}^i d\mathbb{A}_i + \frac{1}{2}\Pi^{ij}(\mathbb{B})\mathbb{A}_i\mathbb{A}_j}$ where $\Pi^{ij}(x) = \frac{x^i * x^j - x^j * x^i}{i\hbar}$ is Kontsevich's deformation of π.



Reference. Abelian and non-abelian BF:
P. Mnev, *Discrete BF theory,* arXiv:0809.1160 (– for M closed),
A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Cellular BV-BFV-BF theory.*
(– with gluing).
1D Chern-Simons: A. Alekseev, P. Mnev, *One-dimensional Chern-Simons theory,* Comm. Math. Phys. 307 1 (2011) 185–227.

1. $\rightarrow$ Corners.
2. Partition function for a "building block" (cell) in interesting examples.
3. Compute cohomology of Ω_∂, e.g. in PSM.
4. More general polarizations, generalized Hitchin's connection.
5. Observables supported on submanifolds.

Main references:

- A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332 2 (2014) 535–603.
- A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Perturbative quantum gauge theories on manifolds with boundary,* arXiv:1507.01221