Quantum BV theories on manifolds with boundary

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Joint work with Alberto S. Cattaneo and Nikolai Reshetikhin

Plan

1. Introduction/motivation I: Chern-Simons theory (perturbative approach).

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4. Abelian BF theory in BV-BFV formalism.

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3. BV-BFV formalism for gauge theories on manifolds with boundary: an outline.
4. Abelian BF theory in BV-BFV formalism.
5. Further examples: Poisson sigma model, cellular models.

Motivation I: Chern-Simons theory

1. Classical Chern-Simons theory:
    - $S = \int_M \operatorname{tr}(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A])$
    - $A \in \operatorname{Conn}(M, G)$.

M an oriented 3-manifold, G a Lie group.

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   - Critical points x_0 should be **non-degenerate**.
   - Γ runs over connected graphs with valence ≥ 3.
   - Φ_Γ is the contraction of $(\partial^2_{x_0} f)^{-1}$ for edges, $\partial^k_{x_0} f$ for k-valent vertex.

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References: Frank W. J. Olver, *Introduction to asymptotics and special functions,* Academic Press, New York, 1974. [leading term]
Pavel Etingof, *Mathematical ideas and notions of quantum field theory,* http://www-math.mit.edu/∼etingof/lect.ps (2002) [Feynman graphs]

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5. Solution: Batalin-Vilkovisky formalism – replaces the integral by one with **non-degenerate** critical points.
6. Output – the perturbative answer (Witten-Axelrod-Singer):

$$Z^{\text{pert}} = \\ = e^{\frac{i}{\hbar}S(A_0)} \cdot \tau(M, A_0)^{\frac{1}{2}} \cdot e^{\frac{\pi i}{4}\psi^{A_0,g}} \cdot \exp\left(\sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} i^{\mathsf{E}+\mathsf{V}} \cdot \Phi_{\Gamma}^{A_0,g}\right) \cdot \\ \cdot e^{ic(\hbar)S_{\text{grav}}(g,\phi)}$$

References:

E. Witten, *Quantum field theory and the Jones polynomial,* Comm. Math. Phys. 121 3 (1989) 351–399.
S. Axelrod and I. M. Singer, *Chern–Simons perturbation theory, I and II,* arXiv:hep-th/9110056 (1991), arXiv:hep-th/9304087 (1993).

Motivation I: Chern-Simons theory – the perturbative answer (**Witten-Axelrod-Singer**):

$$Z^{\mathrm{pert}}(M,G,A_0,\hbar,\varphi)=$$

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- M is **closed**, A_0 is an **acyclic** flat connection.
- $\tau(M,A_0)$ – Reidemeister-Ray-Singer torsion.

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- M is **closed**, A_0 is an **acyclic** flat connection.
- ψ – the Atiyah-Patodi-Singer eta invariant of $L_- = d_E * + * d_E$ on $\Omega^{\mathrm{odd}}(M,E)$. E the flat vector bundle determined by A_0.

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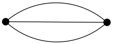
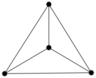
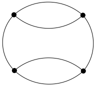
- $\Gamma\in\{$  ,  ,  $,\cdots\}$ – connected 3-valent,

$$\Phi_\Gamma=\int_{\mathrm{Conf}_{\mathsf{V}}(M)}\prod_{\mathrm{edges}\,e}\eta(x_{e_{\mathrm{in}}},x_{e_{\mathrm{out}}})$$

Here $\eta\in\Omega^2(\mathrm{Conf}_2(M),E\boxtimes E)$ is the **propagator** – the integral kernel of d_E^*/Δ_E.

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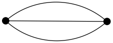
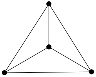
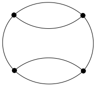
- $\Gamma\in\{$  ,  ,  $,\cdots\}$ – connected 3-valent,

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- g – an arbitrary Riemannian metric, φ – framing of M, $c(\hbar)\in\mathbb{C}[[\hbar]]$ a universal element.

Motivation II: calculating partition functions by cut/paste.

Idea:

$Z\Bigg($  $\Bigg) = \Bigg\langle Z\Bigg($  $\Bigg), Z\Bigg($  $\Bigg)\Bigg\rangle$

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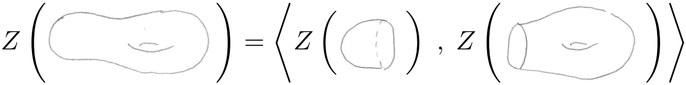
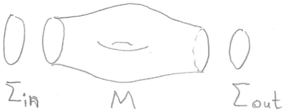
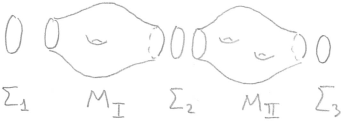


Functorial description (Atiyah-Segal):

Closed $(n-1)$-manifold Σ	$\mathcal{H}_\Sigma$
n-cobordism M  	Partition function $Z_M : \mathcal{H}_{\Sigma_{\mathrm{in}}} \to \mathcal{H}_{\Sigma_{\mathrm{out}}}$
Gluing  	Composition $Z_{M_I \cup M_{II}} = Z_{M_{II}} \circ Z_{M_I}$

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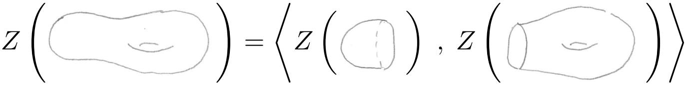

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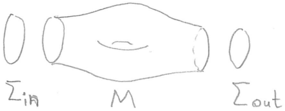
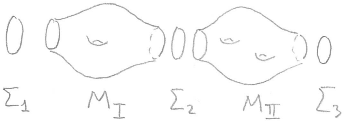
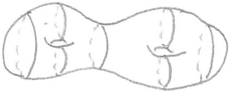
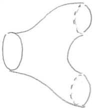
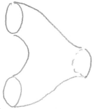
n-cobordism M 	Partition function $Z_M : \mathcal{H}_{\Sigma_{\mathrm{in}}} \to \mathcal{H}_{\Sigma_{\mathrm{out}}}$
Gluing 	Composition $Z_{M_I \cup M_{II}} = Z_{M_{II}} \circ Z_{M_I}$

Atiyah: TQFT is a functor of monoidal categories $(\mathrm{Cob}_n, \sqcup) \to (\mathrm{Vect}_{\mathbb{C}}, \otimes)$.

Example: 2D TQFT

Z ()

can be expressed in terms of building blocks:

1. Z () $: \mathbb{C} \to \mathcal{H}_{S^1}$
2. Z () $: \mathcal{H}_{S^1} \to \mathbb{C}$
3. Z () $: \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1}$
4. Z () $: \mathcal{H}_{S^1} \to \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1}$

– Universal local building blocks for 2D TQFT!

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Building blocks: balls with stratified boundary (cells)

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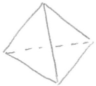
Extension of Atiyah's axioms to gluing with corners: extended TQFT (Baez-Dolan-Lurie).
Example: Turaev-Viro 3D state-sum model.

building block - 3-simplex 	q6j-symbol
gluing	sum over spins on edges

Problems:

- Very few examples!
- Some natural examples do not fit into Atiyah axiomatics.

Goal:

- Construct TQFT with corners and gluing out of perturbative path integrals for diffeomorphism-invariant action functionals.
- Investigate interesting examples.

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Note: $\{S, S\}_\omega = 0$.

BV-BFV formalism for gauge theories on manifolds with boundary

Reference: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332 2 (2014) 535–603.

For M with boundary:

$$\begin{tikzcd}
M \arrow[r] & (\mathcal{F}, \arrow[d, "\pi"] & \omega, & Q, \arrow[d, "\pi_*"] & S) & \text{– space of fields} \\
\partial M \arrow[r] & (\mathcal{F}_\partial, & \omega_\partial = \delta\alpha_\partial, & Q_\partial, & S_\partial) & \text{– phase space}
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Subscripts = "ghost numbers".

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Relations: $Q_\partial^2 = 0,\ \iota_{Q_\partial}\omega_\partial = \delta S_\partial;\qquad Q^2 = 0,\ \boxed{\iota_Q\omega = \delta S + \pi^*\alpha_\partial}$.

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Gluing:

$$M_I \cup_\Sigma M_{II} \to \mathcal{F}_{M_I} \times_{\mathcal{F}_\Sigma} \mathcal{F}_{M_{II}}$$

BV-BFV formalism for gauge theories on manifolds with boundary

Reference: A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332 2 (2014) 535–603.

For M with boundary:

$$\begin{tikzcd}
M \arrow[r] & (\mathcal{F}, \arrow[d, "\pi"] & \omega, & Q, \arrow[d, "\pi_*"] & S) & \text{– space of fields} \\
\partial M \arrow[r] & (\mathcal{F}_\partial, & \omega_\partial = \delta\alpha_\partial, & Q_\partial, & S_\partial) & \text{– phase space}
\end{tikzcd}$$

Relations: $Q_\partial^2 = 0,\ \iota_{Q_\partial}\omega_\partial = \delta S_\partial;\qquad Q^2 = 0,\ \boxed{\iota_Q\omega = \delta S + \pi^*\alpha_\partial}.$

⇒CME: $\frac{1}{2}\iota_Q\iota_Q\omega = \pi^* S_\partial$

Gluing:

$$M_I \cup_\Sigma M_{II} \to \mathcal{F}_{M_I} \times_{\mathcal{F}_\Sigma} \mathcal{F}_{M_{II}}$$

This picture extends to higher-codimension strata!

Example: abelian Chern-Simons theory, $\dim M = 3$.

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$$\begin{tikzcd}
M \arrow[r, long] & (\Omega^\bullet(M)[1], \arrow[d, "\pi\colon \mathcal{A}\mapsto\mathcal{A}|_\partial"] & \omega, & Q, \arrow[d, "\pi_*"] & S) \\
\partial M \arrow[r, long] & (\Omega^\bullet(\partial M)[1], & \omega_\partial = \delta\alpha_\partial, & Q_\partial, & S_\partial)
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Superfield $\mathcal{A} = \underbrace{c}_{\text{ghost},1} + \underbrace{A}_{\text{classical field},0} + \underbrace{A^{+}_{-1} + c^{+}_{-2}}_{\text{antifields}}$

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Euler-Lagrange moduli spaces:

$$\begin{tikzcd}
M \arrow[r] & H^\bullet(M)[1] \arrow[d, "\iota^*"'] \\
\partial M \arrow[r] & H^\bullet(\partial M)[1]
\end{tikzcd}$$

Quantum BV-BFV formalism.

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Reminder: In Darboux coordinates (x^i, ξ_i) on $\mathcal{F}_{\mathrm{res}}$,

$$\Delta_{\mathrm{res}} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}$$

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$*_\Sigma$ — pairing of states in $\mathcal{H}_\Sigma$,
P_* — BV pushforward (fiber BV integral) for
$\mathcal{F}^{M_I}_{\mathrm{res}} \times \mathcal{F}^{M_{II}}_{\mathrm{res}} \xrightarrow{P} \mathcal{F}^{M_I \cup_\Sigma M_{II}}_{\mathrm{res}}$

Aside: BV pushforward.
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Theorem

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Reference: P. Mnev, *Discrete BF theory*, arXiv:0809.1160

Quantization

Choose $p : \mathcal{F}_\partial \to \mathcal{B}$ Lagrangian fibration, $\alpha_\partial|_{p^{-1}(b)} = 0$.

$$\boxed{\mathcal{H}_\partial = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B})}, \ \Omega_\partial = \widehat{S_\partial} \quad \in \mathrm{End}(\mathcal{H}_\partial)_1.$$

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Partition function:

$$Z_M(b) = \int_{\mathcal{L}\subset\mathcal{F}_b} e^{\frac{i}{\hbar}S}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B})$$

$\mathcal{L} \subset \mathcal{F}_b$ gauge-fixing Lagrangian.
Problem: Z_M may be ill-defined due to zero-modes.

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Solution: Split $\mathcal{F}_b = \mathcal{F}_{\mathrm{res}} \times \widetilde{\mathcal{F}} \ni (\phi_{\mathrm{res}}, \widetilde{\phi})$. Partition function:

$$Z_M(b, \phi_{\mathrm{res}}) = \int_{\mathcal{L}\subset\widetilde{\mathcal{F}}} e^{\frac{i}{\hbar}S(b,\phi_{\mathrm{res}},\widetilde{\phi})}, \qquad Z_M \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}) \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{\mathrm{res}})$$

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$$\mathcal{F}_{\mathrm{res}} \xrightarrow{P} \mathcal{F}'_{\mathrm{res}} \qquad \Rightarrow \qquad Z'_M = P_* Z_M$$

Abelian BF theory: the continuum model.

Input:

- M a closed oriented n-manifold M.
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Space of BV fields: $\mathcal{F} = \Omega^\bullet(M, E)[1] \oplus \Omega^\bullet(M, E^*)[n-2] \quad \ni (A, B)$

Action: $S = \int_M \langle B, d_E A\rangle$.

Reference: A. S. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants,* Lett. Math. Phys. 2, 3 (1978) 247–252.

A. S. Schwarz: For M closed and E acyclic, the partition function is the R-torsion $\tau(M, E) \in \mathbb{R}$.

Result, C-M-R arXiv:1507.01221

For M closed, E possibly non-acyclic,
$\mathcal{F}_{\text{res}} = H^\bullet(M,E)[1] \oplus H^\bullet(M,E^*)[n-2]$ and

$$Z_M = \xi \cdot \tau(M,E)$$

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$$\xi = (2\pi\hbar)^{\sum_{k=0}^{n}(-\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot\dim H^k(M,E)} \cdot (e^{-\frac{\pi i}{2}}\hbar)^{\sum_{k=0}^{n}(\frac{1}{4}-\frac{1}{2}k(-1)^k)\cdot\dim H^k(M,E)}$$

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In particular Z_M contains a $\mathrm{mod} 16$ phase $e^{\frac{2\pi i}{16}s}$ with
$s = \sum_{k=0}^{n}(-1+2k(-1)^k)\cdot \dim H^k(M,E)$.

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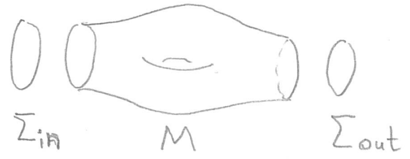
For M with boundary, E possibly non-acyclic,



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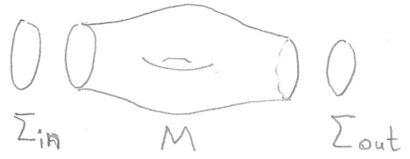

$$Z_M = \xi \cdot \tau(M, \Sigma_{\mathrm{in}}; E) \cdot$$

$$\cdot \exp \frac{i}{\hbar} \left(\int_{\Sigma_{\mathrm{out}}} \mathbb{B}\mathsf{a} + \int_{\Sigma_{\mathrm{in}}} \mathsf{b}\mathbb{A} - \int_{\Sigma_{\mathrm{out}} \times \Sigma_{\mathrm{in}} \ \ni (x,y)} \mathbb{B}(x)\eta(x,y)\mathbb{A}(y) \right)$$

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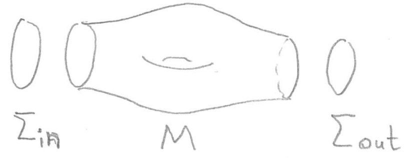

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Where: $\mathcal{F}_{\text{res}} = H^\bullet(M, \Sigma_{\text{in}}; E)[1] \oplus H^\bullet(M, \Sigma_{\text{out}}; E^*)[n-2] \ \ni (\mathsf{a}, \mathsf{b})$

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$$\cdot \exp \frac{i}{\hbar} \left(\int_{\Sigma_{\rm out}} \mathbb{B}\mathsf{a} + \int_{\Sigma_{\rm in}} \mathsf{b}\mathbb{A} - \int_{\Sigma_{\rm out} \times \Sigma_{\rm in} \ni (x,y)} \mathbb{B}(x)\eta(x,y)\mathbb{A}(y) \right)$$

Where: $\mathcal{B} = \Omega^\bullet(\Sigma_{\rm in})[1] \oplus \Omega^\bullet(\Sigma_{\rm out})[n-2] \ni (\mathbb{A}, \mathbb{B})$

$$\mathcal{H}_\Sigma = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}) \quad \ni \sum_{k,l \geq 0} \int_{\mathrm{Conf}_k(\Sigma_{\rm in}) \times \mathrm{Conf}_l(\Sigma_{\rm out})}$$

$$\Psi(x_1, \ldots, x_k; y_1, \ldots, y_l) \mathbb{A}(x_1) \cdots \mathbb{A}(x_k) \mathbb{B}(y_1) \cdots \mathbb{B}(y_l)$$

Result, C-M-R arXiv:1507.01221

For M with boundary, E possibly non-acyclic,

$$Z_M = \xi \cdot \tau(M, \Sigma_{\text{in}}; E) \cdot$$
$$\cdot \exp \frac{i}{\hbar} \left(\int_{\Sigma_{\text{out}}} \mathbb{B}\mathsf{a} + \int_{\Sigma_{\text{in}}} \mathsf{b}\mathbb{A} - \int_{\Sigma_{\text{out}} \times \Sigma_{\text{in}} \ni (x,y)} \mathbb{B}(x)\eta(x,y)\mathbb{A}(y) \right)$$

Where: ξ as before (but for relative cohomology),

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Where: τ - relative R-torsion,

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Where: $\eta \in \Omega^{n-1}(\mathrm{Conf}_2(M), E \boxtimes E^*)$ – propagator, i.e. $\alpha \mapsto \int_{M\ni y} \eta(x,y)\alpha(y)$ is a chain contraction from $\Omega^\bullet(M, \Sigma_{\mathrm{in}}; E)$ to $H^\bullet(M, \Sigma_{\mathrm{in}}; E)$.

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This result satisfies:

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- change of η shifts Z_M by $\left(\frac{i}{\hbar}\Omega_\partial - i\hbar\Delta_{\text{res}}\right)$-exact term.

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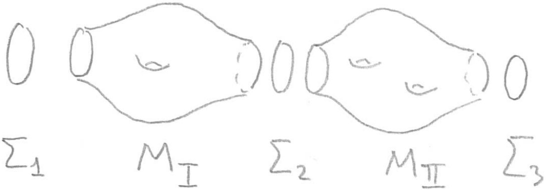
BFV operator: $\Omega_\partial = -i\hbar \left(\int_{\Sigma_{\text{out}}} d_E \mathbb{B} \frac{\delta}{\delta \mathbb{B}} + \int_{\Sigma_{\text{in}}} d_E \mathbb{A} \frac{\delta}{\delta \mathbb{A}} \right)$

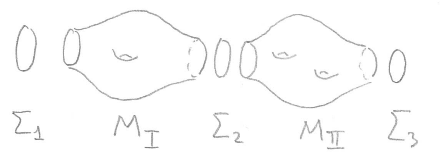
Gluing



in two steps:

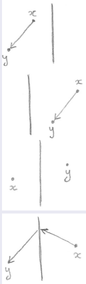
1. $\widetilde{Z}_M = \int_{\mathbb{A}_2,\mathbb{B}_2} Z_{M_{II}}(\mathbb{B}_3, \mathbb{A}_2; \mathsf{a}_{II}, \mathsf{b}_{II}) \cdot e^{\frac{i}{\hbar}\int_{\Sigma_2} \mathbb{B}_2 \mathbb{A}_2} \cdot Z_{M_I}(\mathbb{B}_2, \mathbb{A}_1; \mathsf{a}_I, \mathsf{b}_I)$.
2. $Z_M = P_* \widetilde{Z}_M$, for $P : \mathcal{F}^I_{\mathrm{res}} \times \mathcal{F}^{II}_{\mathrm{res}} \to \mathcal{F}_{\mathrm{res}}$.

Result, C-M-R arXiv:1507.01221



η_I, η_{II} – propagators on M_I, M_{II}.
Assume $H^\bullet(M,\Sigma_1) = H^\bullet(M_I,\Sigma_1) \oplus H^\bullet(M_{II},\Sigma_2)$.
Then the glued propagator on M is:

$$\eta(x,y) = \begin{cases} \eta_I(x,y) & \text{if } x,y \in M_I \\ \eta_{II}(x,y) & \text{if } x,y \in M_{II} \\ 0 & \text{if } x \in M_I, y \in M_{II} \\ \boxed{\int_{z\in\Sigma_2} \eta_{II}(x,z)\eta_I(z,y)} & \text{if } x \in M_{II}, y \in M_I \end{cases}$$



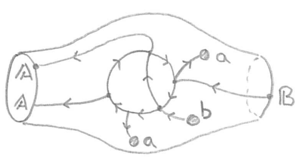
Example: Poisson sigma model, $n = 2$.

Action: $S = \int_M \langle B, dA \rangle + \frac{1}{2} \langle \pi(B), A \otimes A \rangle$

$\pi = \sum_{ij} \pi^{ij}(u) \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial u^j}$ Poisson bivector on $\mathbb{R}^m$.

Result, C-M-R arXiv:1507.01221

$$Z_M = \xi \cdot \tau \cdot \exp \frac{i}{\hbar} \sum_{\text{graphs}}$$



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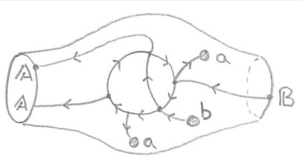

Z_M satisfies:

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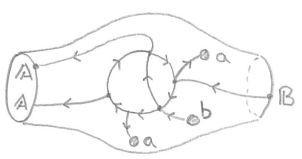

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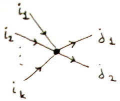
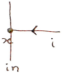
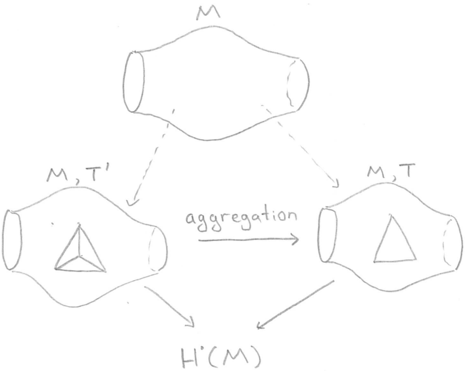
Ω_∂ = standard-ordering quantization ($\mathbb{B} \mapsto -i\hbar \frac{\delta}{\delta \mathbb{A}}$ on Σ_{in}, $\mathbb{A} \mapsto -i\hbar \frac{\delta}{\delta \mathbb{B}}$ on Σ_{out}) of $\boxed{\int_\partial \mathbb{B}^i d\mathbb{A}_i + \frac{1}{2} \Pi^{ij}(\mathbb{B}) \mathbb{A}_i \mathbb{A}_j}$ where $\Pi^{ij}(u) = \frac{u^i * u^j - u^j * u^i}{i\hbar}$ is Kontsevich's deformation of π.

Rules for calculating Φ_Γ ("Feynman rules").

Decorate half-edges by $i \in \{1, \ldots, m\}$, put internal vertices to $z_1 \ldots, z_p \in M$, boundary in-vertices to $x_1, \ldots, x_k \in \Sigma_{\text{in}}$, boundary out-vertices to $y_1, \ldots, y_l \in \Sigma_{\text{out}}$. Assign:

	$\partial_{i_1} \cdots \partial_{i_k}\big\|_0 \pi^{j_1 j_2}$
	a_i
	b^i
	$A_i(x)$
	$B^i(y)$
	$\eta(z, z')$

Sum over i-labels, integrate over positions of vertices.



Reference. Abelian and non-abelian BF:
P. Mnev, *Discrete BF theory,* arXiv:0809.1160 (– for M closed),
A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Cellular BV-BFV-BF theory.*
(– with gluing).
1D Chern-Simons: A. Alekseev, P. Mnev, *One-dimensional Chern-Simons theory,* Comm. Math. Phys. 307 1 (2011) 185–227.

Example: abelian BF theory on a cobordism with a cell decomposition.

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- M an n-cobordism, T a cell decomposition. $T^\vee$ – dual decomposition.

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- Quantization – as in continuum case, but replacing differential forms by cellular cochains. R-torsion appears as a measure-theoretic integral rather than regularized ∞-dimensional integral.

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- Consistent with BV pushforwards along cellular aggregations $T' \to T$.

Further program

1. → Corners.
2. Partition function for a "building block" (cell) in interesting examples.
3. Compute cohomology of Ω_∂, e.g. in PSM.
4. More general polarizations, generalized Hitchin's connection.
5. Chern-Simons theory in BV-BFV formalism: extension of Axelrod-Singer's treatment to 3-manifolds with boundary/corners.
    - Comparison with Witten-Reshetikhin-Turaev non-perturbative answers.
    - Prove the conjecture that $k \to \infty$ asymptotics of the RT invariant on a closed 3-manifold is given by Axelrod-Singer expansion.
6. Observables supported on submanifolds.

Main references

- A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Classical BV theories on manifolds with boundary,* Comm. Math. Phys. 332 2 (2014) 535–603.
- A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Perturbative quantum gauge theories on manifolds with boundary,* arXiv:1507.01221

Cellular realizations:

- P. Mnev, *Discrete BF theory,* arXiv:0809.1160.
- A. Alekseev, P. Mnev, *One-dimensional Chern-Simons theory,* Comm. Math. Phys. 307 1 (2011) 185–227.
- A. S. Cattaneo, P. Mnev, N. Reshetikhin, *Cellular BV-BFV-BF theory,* in preparation.