Talk for ND Grad Student Seminar
(April 19, 2021)

Topological quantum mechanics, Stasheff's associahedra and homotopy transfer

① Homotopy transfer

Let $(V^\bullet, \underset{\uparrow\ \text{differential}}{d}, \underset{\uparrow\ \text{multiplication}}{m})$ be a differential graded algebra

$d: V^\bullet \to V^{\bullet+1}$
$m: V^k \otimes V^\ell \to V^{k+\ell}$

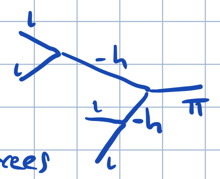
s.t.

$d^2 = 0$
$d(x\cdot y) = dx\cdot y + (-1)^{|x|} x\cdot dy$
$(x\cdot y)\cdot z = x\cdot(y\cdot z)$

And let $V^\bullet = \underbrace{V'^\bullet}_{\text{def. retract}} \oplus \underbrace{V''^\bullet}_{\text{acyclic}}$ - splitting of complexes.

Then $V'^\bullet$ inherits an A_∞ algebra structure, $m_n: V'^{\otimes n} \to V'$
$m_1 = d_{V'}$

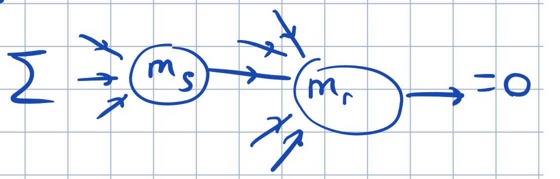
Given by $m_n = \sum_{\substack{\text{binary rooted trees}\\ \text{with } n \text{ leaves}}}$



$x_1 \otimes \dots \otimes x_4 \longmapsto \pi\Big(\big(-h(\iota(x_1)\cdot\iota(x_2))\big)\cdot\big(-h\,(\iota(x_3)\cdot\iota(x_4))\big)\Big)$

where $\iota: V' \hookrightarrow V$ inclusion
$\pi: V \twoheadrightarrow V'$ projection
$h: V \to V$ chain homotopy : $dh + hd = id - \iota\circ\pi$

- $\{m_n\}$ satisfy the "A_∞ relations" $\sum_{\substack{r+s=n+1\\ 1\le i\le r}} m_r \circ_i m_s = 0$



Or: $(\hat m_1 + \hat m_2 + \hat m_3 + \dots)^2 = 0$
$\hat m_k : TV' \to TV'$ - extension of m_k to a coderivation of TV'

Ex: $V = \Omega^\bullet(\underbrace{M}_{\text{cpt Riem mfd}})$

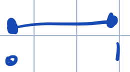
$V' = \mathrm{Harm}^\bullet(M) \simeq H^\bullet(M)$, $V'' = \Omega_{ex} \oplus \Omega_{coex}$;

$h = \begin{cases} 0 & \text{on Harm} \\ d^*\Delta^{-1} & \text{on } V'' \end{cases}$

Then m_n are "Massey operations" on $H(M)$.
- encode $\mathbb{Q}\otimes\pi_k(M)$ if $\pi_1(M) = 0$.

E.g. $M = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} \Big/ \left\{ \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ has vanishing $m_2 \quad H^1 \otimes H^1 \to H^2$
but nonvanishing $m_3 \quad H^1 \otimes H^1 \otimes H^1 \to H^2$

Heisenberg nilmanifold

Ex: $V = \Omega^\bullet([0,1])$
$V' = C^\bullet($  $)$ $\rightsquigarrow$

cell cochains

$$m_{n+1}(\underbrace{e_{01}, \dots, e_{01}}_{k}, e_1, \underbrace{e_{01}, \dots, e_{01}}_{n-k}) = (-1)^k \binom{n}{k} \frac{B_n}{n!} e_{01}$$

$$m_{n+1}(e_{01} \cdots e_{01}, e_0, e_{01}, \dots, e_{01}) = -(-1)^k \binom{n}{k} \frac{B_n}{n!} e_{01}$$

$$m_2(e_0, e_0) = e_0 \quad , \quad m_2(e_1, e_1) = e_1$$

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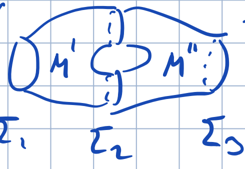
② Atiyah's axioms of n-TQFT

- $(n-1)$-dim closed mfd $\Sigma \longrightarrow V_\Sigma$ - vector space
- n-cobordism $\Sigma_1 \xrightarrow{M} \Sigma_2 \longrightarrow I_M : V_{\Sigma_1} \to V_{\Sigma_2}$ - linear map

- $\varphi \in \mathrm{Diff}(\Sigma) \longrightarrow \rho_\varphi : V_\Sigma \to V_\Sigma$

Axioms: • multiplicativity: $\amalg \longrightarrow \otimes$

• gluing:

$I\Big($  $\Big) = I_{M''} \circ I_{M'}$

• normalization: $V_\emptyset = \mathbb{k} \quad , \quad I_{\Sigma \times [0,1]} = \mathrm{id}_{V_\Sigma}$

• Diff-equivariance:
for $\Phi : M \to \tilde{M}$ diffeo ,

$$\begin{tikzcd} V_{\Sigma_1} \arrow[r, "I_M"] \arrow[d, "\rho(\Phi|_{in})"'] & V_{\Sigma_2} \arrow[d, "\rho(\Phi|_{out})"] \\ V_{\tilde{\Sigma}_1} \arrow[r, "I_{\tilde{M}}"] & V_{\tilde{\Sigma}_2} \end{tikzcd}$$

In particular, $(V, I) : \mathrm{Cob}_n \to \mathrm{Vect}$ is a functor of monoidal categories

- Segal's version: $I_M = I_{M,\gamma}$ depends on local geom. structure $\gamma \in \mathrm{Geom}_M$ on M.

E.g. γ is a
- Riem. metrix
- complex structure
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Ex (quantum mechanics): $n=1$, $pt \to V$ (space of states), set $\mathrm{Geom}_{\vdash\!\dashv} = \left\{ \begin{smallmatrix}\text{Riem.}\\ \text{metrics on } I\end{smallmatrix} \right\}$

Then $I(\underset{\substack{t>0 \\ \uparrow \\ \text{length}}}{\vdash\!\!\!\longrightarrow\!\!\!\dashv}) \in \mathrm{End}(V)$, $\quad \underset{t_1}{\vdash\!\!\!\longrightarrow\!\!\!\dashv} \cup \underset{t_2}{\vdash\!\!\!\longrightarrow\!\!\!\dashv} \overset{}{=} \underset{t_1+t_2}{\vdash\!\!\!\longrightarrow\!\!\!\dashv} \quad \overset{(*)}{\Rightarrow} I_{t_1+t_2} = I_{t_1} \circ I_{t_2}$

- semi-group law

Solution of (*): $I_t = e^{-tH}$, $\quad H \in \mathrm{End}(V)$

non-negative operator - Hamiltonian

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① $\underset{\substack{\uparrow \\ \text{"higher"}}}{\text{H}}$**TQFT** :

(or - derived
- cohomological
- cochain level)

$\Sigma \to (V^\bullet, Q)$ - chain complex,

$M \to I_M \in \Omega^\bullet(\mathrm{Geom}_M) \otimes \mathrm{Hom}(V_{\Sigma_1}, V_{\Sigma_2})$

satisfying $\boxed{(d_{\mathrm{Geom}} - Q_{\Sigma_1} + Q_{\Sigma_2})\, I_M = 0}$

Rem $I_M^{(0)}$ ← degree in $\Omega^\bullet_{\mathrm{Geom}}$

is a chain map $V_{\Sigma_1} \to V_{\Sigma_2}$;

induced map $[I_M^{(0)}] : H_Q(V_{\Sigma_1}) \to H_Q(V_{\Sigma_2})$ is (locally) constant on Geom.

- (ordinary) Atiyah's TQFT.

Ex (HTQM): $n=1$. $\quad pt \to V^\bullet$, $\quad Q: V^\bullet \to V^{\bullet+1}$ differential

$G: V^\bullet \to V^{\bullet-1}$ "non-normalized chain homotopy"

s.t. $H = QG + GQ$ is non-negative

$$I(\underset{t,dt}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}) = \boxed{e^{-tH - dt\, G}} \in \Omega^\bullet(\underbrace{\mathbb{R}_+}_{\substack{\text{moduli space of}\\ \text{metric intervals}}}) \otimes \mathrm{End}(V)$$

$$\overset{\shortparallel}{e^{-tH}(1 - dt\, G)}$$

recovered from gluing axiom and $I(\underset{t,dt}{\vdash\!\dashv}) = 1 - tH - dt\, G + O(t^2)$

short interval

(*)
- $(d_t + [Q, -])\, I = 0$ is satisfied

Ex: a) $V^\bullet = \Omega^\bullet(\underset{\text{cpt., Riem.}}{M})$, $Q = d$, $G = d^*$ $\Rightarrow H = \Delta \circlearrowright \Omega^\bullet(M)$

Rem $I|_{t=0} = id$, $I|_{t=+\infty} = P_{\text{Harm}}$, $-\int_0^\infty I = \begin{cases} 0 & \text{on Harm} \\ d^*\Delta^{-1} & \end{cases}$

$\left((*) \Rightarrow \left[Q, -\int_{t_0}^{t_1} I\right] = I|_{t_0} - I|_{t_1} \right)$

= chain homotopy between id and P_{harm}.

b) $V^\bullet = \Omega^\bullet(M)$, $Q = d$, $G = \iota_{\underset{\text{vector field}}{v}}$ $\Rightarrow H = \mathcal{L}_v$

c) $(V^\bullet, \underset{d}{Q})$ a chain complex with a retraction
$$\begin{tikzcd} V \arrow[loop right, "h"] \arrow[d, shift left, "\pi"] \\ V' \arrow[u, shift left, "\iota"] \end{tikzcd}$$
. Set $G = h$

$\Rightarrow H = \begin{cases} 0 & \text{on } \iota(V') \\ id & \text{on } V'' = \ker \pi \end{cases}$

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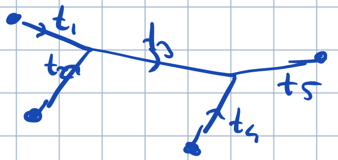
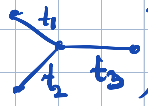
④ HTQM on metric trees

Let (V, d, m) be a DGA and let
$$\begin{tikzcd} V \arrow[loop right, "h"] \arrow[d, shift left, "\pi"] \\ V' \arrow[u, shift left, "\iota"] \end{tikzcd}$$
be a retraction

instead of Cob_1, consider binary, rooted trees with metric on edges

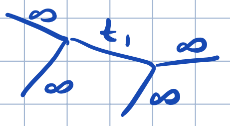


$I_{\underset{\text{tree}}{T}} \in \Omega^\bullet(\mathbb{R}_+^{\#\text{edges}}) \otimes \text{Hom}(V^{\otimes \text{leaves}}, V)$

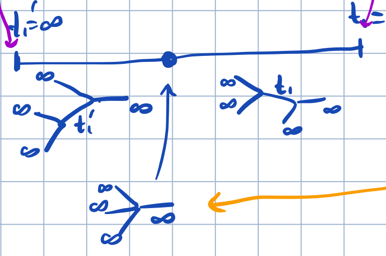
Eg. I()$(x_1 \otimes x_2) = e^{-t_3 H - dt_3 G} \circ m\left(e^{-t_1 H - dt_1 G}(x_1), e^{-t_2 H - dt_2 G}(x_2)\right)$

with $G = h$, $H = P_{V'}$

"pre-amplitude": $I_n = I\Big|_{\substack{\text{trees with } \infty\text{-long ends} \\ \text{with } n \text{ leaves}}} \in \Omega^\bullet(\underset{\text{moduli of metric trees with } \infty\text{-long ends.}}{MTrees_n^\infty}) \otimes \underbrace{\text{Hom}(V^{\otimes n}, V)}_{\downarrow\ \text{Hom}(V'^{\otimes n}, V')}$



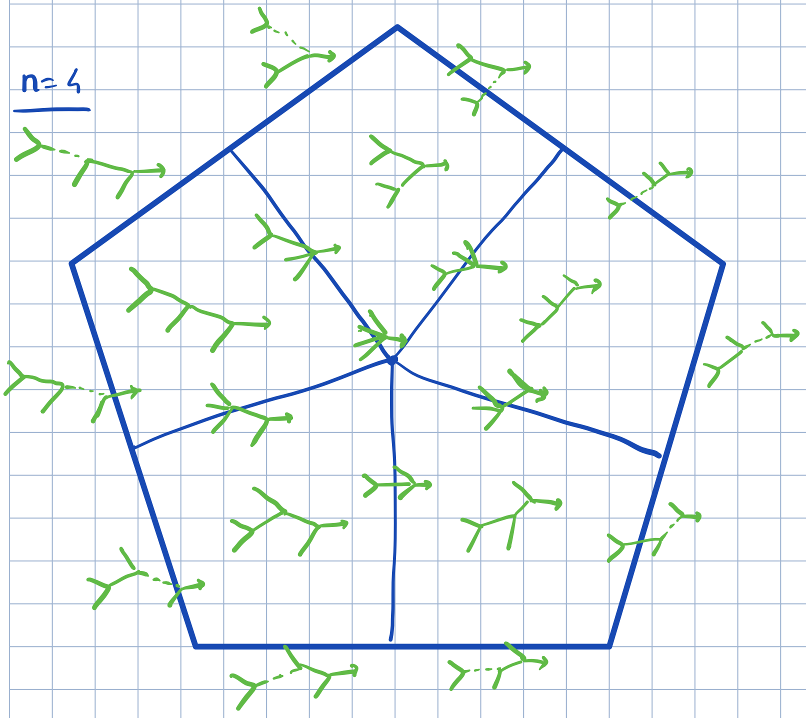
n=3 $MTrees_3$:



I_3 is
- $(d_{MT} + d_{V'})$-closed on $MTrees_3$ ↙ using $[Q, m] = 0$ - Leibniz
- continuous at (due to associativity of m)

thus: $d_{MT} I^{(0)} + d_{V'} I^{(1)} = 0 \xrightarrow{\int_{MTrees}} \underbrace{\int_{\partial MTrees} I^{(0)}}_{\overset{\|}{m_2 \circ_1 m_2 + m_2 \circ_2 m_2}} + d_{V'} \Big(- \underbrace{\int_{MTrees} I^{(1)}}_{m_3} \Big)$ – associativity relation

n=4



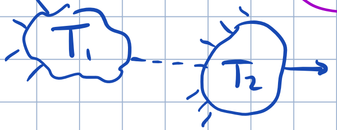
$MTrees_4 \overset{\|}{} K_4$ – Stasheff's associahedron

$$d_{MT} I^{(1)} + d_{V'} I^{(2)} = 0$$

$$\rightsquigarrow$$

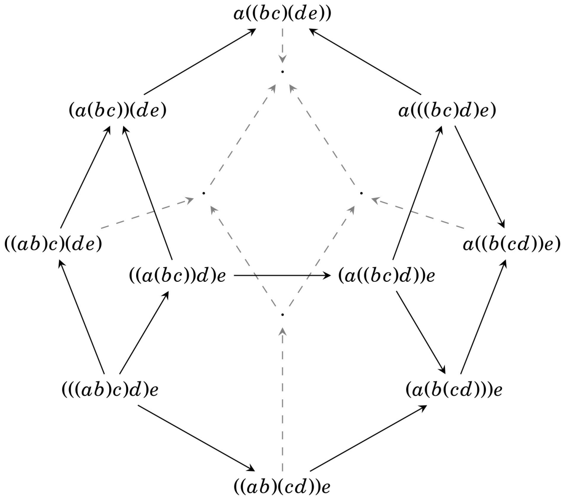
$$\underbrace{\int_{\partial MTrees_4} I^{(1)}}_{m_2 \circ m_3 + m_3 \circ m_2} + d_{V'} \underbrace{\int_{MTrees_4} I^{(2)}}_{m_4} = 0$$

Generally: $\underbrace{\int_{\partial MTrees_n} I^{(n-3)}}_{} + \underbrace{d_{V'} \underbrace{\int_{MTrees_n} I^{(n-2)}}_{m_n}}_{d_{V'} \circ m_n + \sum_{i=1}^{n} m_n \circ_i d_{V'}} = 0$ – A_∞-relation for $(V', m_1 = d_{V'}, \underbrace{m_2, m_3, \ldots}_{\text{HTQM amplitudes}})$

$\sum_{T_1, T_2} \int_{MT_1} \int_{MT_2}$ 

$$\overset{\|}{\sum_{\substack{r+s=n \\ 1 \le i \le r+1 \\ r,s \ge 2}}} m_{r+1} \circ_i m_s$$

$I_{T_1 - T_2} = I_{T_2} \circ_i I_{T_1}$ – factorization of the pre-amplitude I on the boundary strata of MTrees



<u>Rem</u> if $\delta \in \mathrm{End}_{+1}(V)$, $[d,\delta]=0$, $[\delta,\delta]=0$,

then $$\delta' := \sum_n \int_{t_1,\dots,t_n} I\Big(\underset{\infty}{\vdash\!\!-}\underset{\delta}{(\,)}\underset{t_1}{-}\underset{\delta}{(\,)}\underset{t_2}{-} \cdots \underset{t_n}{-}\underset{\delta}{(\,)}\underset{\infty}{-\!\!\dashv} \Big) \in \mathrm{End}_{+1}(V')$$

satisfies $(d_{V'}+\delta')^2=0$

δ' - induced perturbation of the differential on V.

(homol. perturbation lemma); $\boxed{H_{d_{V'}+\delta'} \simeq H_{d+\delta}}$

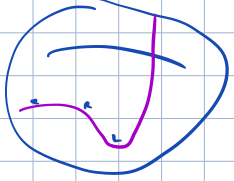
<u>Further application</u>: Gromov-Witten invariants

$$I \in \Omega^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathrm{Hom}(H^\bullet(\underset{\substack{\uparrow\\ \text{target Kähler mfd}}}{X})^{\otimes n}, \mathbb{R})$$ - closed forms on $\mathcal{M}_{g,n}$

- factorizing on compactification strata

$$\Big(\int_{\overline{\mathcal{M}}_{g,n}} I\Big)(\delta_{\alpha_1},\dots,\delta_{\alpha_n}) = \#\underset{\text{of genus } g}{\text{curves}} \text{ in } X \text{ passing through } \overset{\text{cycles}}{\alpha_1,\dots,\alpha_n}$$

Keel's relation for $H^\bullet(\mathcal{M}_{g,n}) \rightsquigarrow$ equation on amplitudes $\int_{\overline{\mathcal{M}}_{g,n}} I$



- "WDVV equation"

- allows one to "predict" GW invariants : $X=\mathbb{C}P^2$

$N_d = \#$ genus 0 curves of degree d through $3d-1$ points:

$N_1 = 1$

$N_2 = 1$

$N_3 = 12$ (Kontsevich-Manin)

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