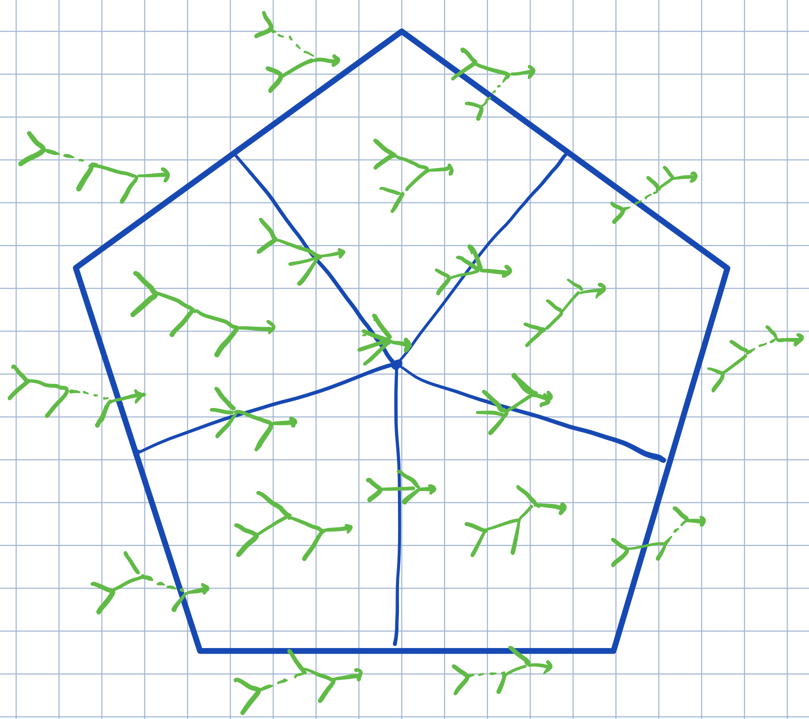
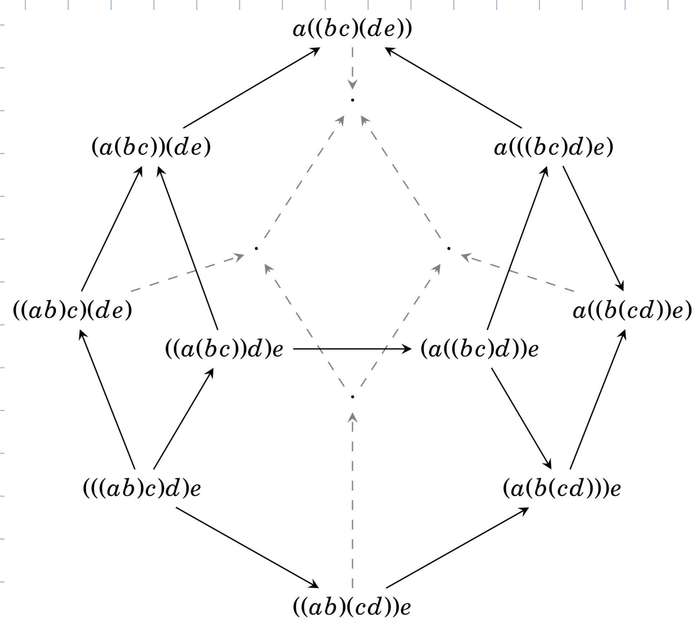




$a((bc)(de))$
$(a(bc))(de)$
$a(((bc)d)e)$
$((ab)c)(de)$
$a((b(cd))e)$
$((a(bc))d)e$
$(a((bc)d))e$
$(((ab)c)d)e$
$(a(b(cd)))e$
$((ab)(cd))e$

I TQM amplitudes & homotopy transfer ← A. Losev "TQFT, homol. algebra and elements of K. Saito's theory of primitive form"

II Gromov-Witten invariants & WDVV ← Kontsevich - Manin '94

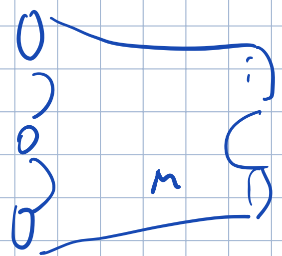
- HTQFT

  ↑ higher cochain-level

Segal's setup

$(\mathcal{H}, Z) : \quad Cob_n^{geom} \to Ch^\bullet$

$\qquad\qquad\qquad\qquad\qquad\qquad \uparrow (\mathcal{H}^\bullet, \underset{\cdots}{Q})$



$\longrightarrow \quad Z \in \underbrace{\Omega^\bullet(\mathrm{Geom}_M)}_{\equiv\, \mathrm{Fun}(\Pi T\, \mathrm{Geom}_M)} \otimes \mathrm{Hom}(\mathcal{H}_{\Sigma_{in}}, \mathcal{H}_{\Sigma_{out}})$

axioms $\quad \sqcup \to \otimes$

$\qquad\qquad \cup \to \circ$

$\qquad$ Diff - equivar

$$(*) \quad \boxed{(d_{\mathrm{Geom}} + Q_{\partial\Sigma}) Z^\bullet = 0} \quad \leftarrow \text{" topological property "}$$

$Z = Z^{(0)} + Z^{(1)} + \cdots$ ← in $\Omega^1(\mathrm{Geom})$

$Q_{\partial\Sigma} Z^{(0)} = 0 \qquad (0)$

$d_{\mathrm{Geom}} Z^{(0)} + Q_{\partial\Sigma} Z^{(1)} = 0 \qquad (1)$

$d_{\mathrm{Geom}} Z^{(0)} = Q_{\partial\Sigma}(\cdots)$

$Z^{(0)}$ - chain map $\mathcal{H}_{\Sigma_{in}} \to \mathcal{H}_{\Sigma_{out}}$

$[Z^{(0)}] : \quad H^\bullet_{Q_{in}}(\mathcal{H}_{in}) \to H^\bullet_{Q_{out}}(\mathcal{H}_{out})$

↑ loc. const on Geom $\qquad$ ↖ usual Atiyah's TQFT

- object of interest "amplitudes" $\quad \int_\gamma Z_M$ ← γ cycle in Geom_M

- (topological) string amplitudes $\quad \mathrm{Geom} = \overline{\mathcal{M}}_{g,n} \quad Z_M \leftarrow$ some top. conf. theory

$\langle G \cdots G \ \mathcal{O}' \cdots \mathcal{O} \rangle$

- solutions to enum. problems
  (A-model)

- by example: K-S formula for HT

  dga
  $\downarrow$
  A_∞ str. on def. retract

- amplitudes satisfy interesting relations

  - A_∞ relation on $\{m_n\}$
  WDVV (associativity eq.)

(Universal) 1d HTQFT (= HTQM)

Geom = metrics on an interval

$\bullet \to \mathcal{H}, Q$

Riem. metrics / diffeo

$$Z(\underset{t}{\longleftarrow\!\!\longrightarrow}) \in \Omega^\bullet(\underbrace{\mathbb{R}_+}_{\text{mod. space of metric interval}}) \otimes \mathrm{End}(\mathcal{H})$$

$Z(t, dt)$

$$p^*(Z_{I_1 \cup I_2}) = Z_{I_1} \circ Z_{I_2}$$

$\circ$ in End
$\otimes$ in $\Omega^\bullet$

$p: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$
$(s, t) \mapsto s+t$

for t small

$$Z(\underset{t}{\vdash\!\dashv}) \overset{!}{=} id - t\underset{\mathrm{End}_0(\mathcal{H})}{\underset{\cap}{H}} - dt \underset{\mathrm{End}_{-1}(\mathcal{H})}{\underset{\cap}{G}} + o(t)$$

$(d_t + [Q, -])Z = 0$

(0) $[Q, H] = 0$

(1) $\boxed{H = [Q, G]}$

$$Z(\vdash\!\!+\!\!+\!\!+\!\!+\!\!+\!\!+\!\!+\!\!+\!\!+\!\!+\!\!+\!\!\dashv) = Z\left(\frac{t}{N}, \frac{dt}{N}\right)^N = \boxed{e^{-tH - dt\, G}}$$

non-normalized chain-homotopy

$$\text{"} \quad e^{-tH}(1 - dt\, G)$$

H = Hamiltonian

- Witten '82

$$[Q_i, Q_j] = H\,\delta_{ij} \qquad i,j = 1,2$$

$$Q_1 = Q + G \qquad Q_2 = i(Q - G)$$

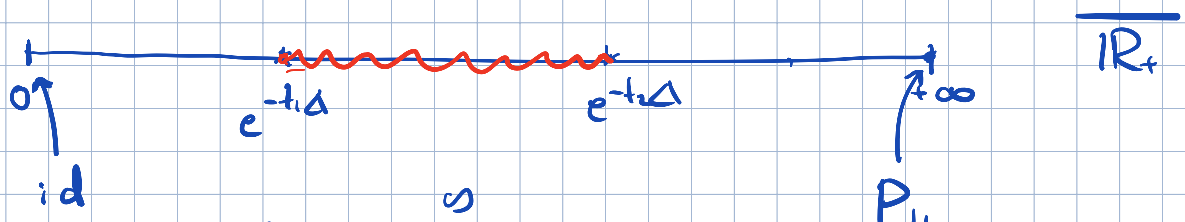
$N=2$ SUSY

Examples: (a) Hodge TQM

$$\mathcal{H} = \Omega^\bullet(\underbrace{M}_{\text{cpt Riem}}) \qquad Q = d \qquad G = d^*$$

$$H = [d, d^*] = \Delta$$

$$Z(t, dt) = e^{-t\Delta}(1 - dt\, d^*)$$



$$\int_{\mathbb{R}_+} Z = \int_0^\infty -dt\, d^* e^{-t\Delta} = \begin{cases} -d^*\Delta^{-1} & \text{on } \Omega_{ex} \oplus \Omega_{coex} \\ 0 & \text{on Harm} \end{cases}$$

$= -h$

chain homotopy

$$\mathrm{id} - P_{Harm} = dh + hd$$

(b)

$$\mathcal{H} = \Omega^\bullet(M), \quad Q = d, \quad G = \iota_v$$

some vector field on M

$$[Q, G] = \mathcal{L}_v$$

$v = \mathrm{grad}\, f$

interpolation between (a) and (b) for $v = \mathrm{grad} f$

$$\boxed{Q = d \qquad G = d^* + \lambda\, \iota_{\mathrm{grad}\, f}}$$

(c) $\mathcal{H} = V \otimes \mathbb{C}[c]$

odd variable

$$Q = c H_V \qquad G = \frac{\partial}{\partial c}$$

$H_V \in \mathrm{End}(V)$

$$H^{tot} = [Q, G] = H_V \otimes \mathrm{id}_{\mathbb{C}[c]}$$

toy model for bosonic string

(d) $(H, Q) = (V^{\bullet}, d)$ ch. complex

$$\begin{tikzcd} V \arrow[loop right, "h"] \arrow[d, bend right, "p"] \\ V' \arrow[u, bend left, "i"] \end{tikzcd}$$ retraction

$$Q = d \qquad G = h$$

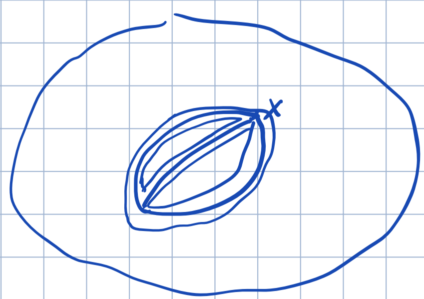
$$H = [d, h] = id - i\circ p$$

$H = \mathrm{Gen}\,Q(\Phi, \iota, P)$

$Q \leftsquigarrow S_{DFU}$

$\overset{\shortparallel}{\Omega}$

<u>Loc. observables</u>



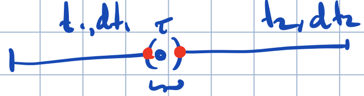
$X \subset M$

$\partial(U_X)$

$\mathcal{O} \in \mathcal{H}_{\partial U_X}$

supported on X

$$\langle \mathcal{O} \rangle = \langle Z_{M - U_X}, \mathcal{O} \rangle_{\mathcal{H}_{\partial U_X}}$$



$\mathcal{O}_\tau \in \mathcal{H}_{\partial U_\tau} \in \mathrm{End}(\mathcal{H})$

$[Q, \mathcal{O}_\tau] = 0$

$(d + Q) Z_\tau = 0$ deformed by observables

fix $\mathcal{O} \in \mathrm{End}_{+1}(\mathcal{H})$

s.t. $[Q, \mathcal{O}] = 0$, $[\mathcal{O}, \mathcal{O}] = 0$

$\mathcal{H}' = \ker H$

$\sum_n \int_{t_1, \dots, t_n \in [0, \infty]^{\times n}}$  $=$

$$= \underbrace{\sum_n P_{\mathcal{H}'} \mathcal{O} (-h) \mathcal{O} (-h) \dots (-h) \mathcal{O} P_{\mathcal{H}'}}_{\text{deformation of the differential } Q_{\mathcal{H}'} \text{ induced from the def. } Q \to Q + \mathcal{O}} = A^{tot}$$

$\leftarrow$ fmla of homol. pert. lemma

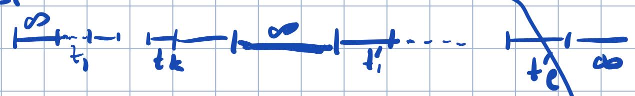
$$\int d Z^{\bullet} + \underbrace{\int [Q, Z^{\bullet}]}_{} = 0$$

$A_n = \int Z$ - amplitude

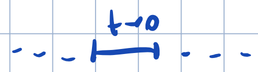
MT ...

$MI_n = [0,\infty)$ $\quad [Q, A_n]$ $\quad MI_n$

$$\int_{\partial MI_n} Z^{\bullet} = \sum_{k+l=n-1} \int Z_n = \int_{k+l=n-1} A_l \circ A_k$$



$MI_k \times MI_l$



doesn't contribute because $\partial^2 = 0$

$Z_l \circ Z_k$

$$\Big(Q + \underbrace{A_1 + A_2 + \dots}\Big)^2 = 0 \quad \sim \quad \big(Q_{H'} + A|_{H'}\big)^2 = 0$$

TQM on trees

$\mathcal{H} = (V, d, \underbrace{m}_{\text{product}})$ $\quad$ dga

observable

Cob_1

$\rightsquigarrow$

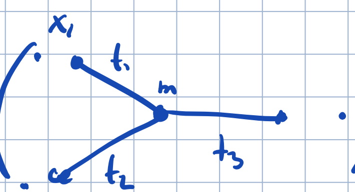
Met Trees

binary rooted trees

fix G

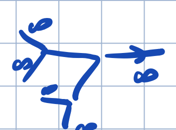
$H = [d, G]$

$V' = \ker H \hookrightarrow V$

$Z\Big($  $\Big) \in \Omega^{\bullet}(\mathbb{R}_+^{\times 3}) \otimes \mathrm{Hom}(V^{\otimes 2}, V)$

$$x_1 \otimes x_2 \longmapsto Z_{t_3, dt_3} \circ m\big(Z_{t_1, dt_1}(x_1) \otimes Z_{t_2, dt_2}(x_2)\big)$$

Consider the restriction of Z to MT_n^{∞}



moduli of trees with ∞-long ends with n leaves

$$Z|_{MT_n^{\infty}} \in \Omega^{\bullet}(MT_n^{\infty}) \otimes \underbrace{\mathrm{Hom}(V^{\otimes n}, V)}_{\mathrm{Hom}(V'^{\otimes n}, V')}$$

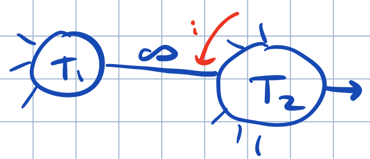
$$\int_{MT_n^{\infty}} Z =: m_n \in \mathrm{Hom}(V'^{\otimes n}, V')$$

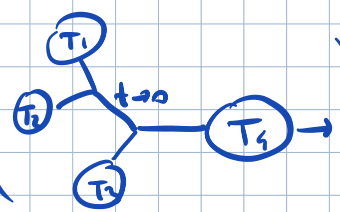
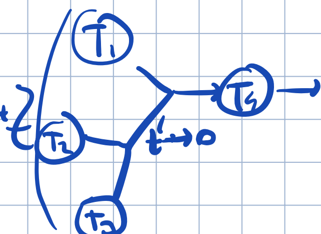
$(V', d_{V'}, m_2, m_3, \dots)$ A_∞-alg

$$\overset{\|}{m_1} \qquad \underbrace{\qquad\qquad}_{\text{amplitudes}}$$

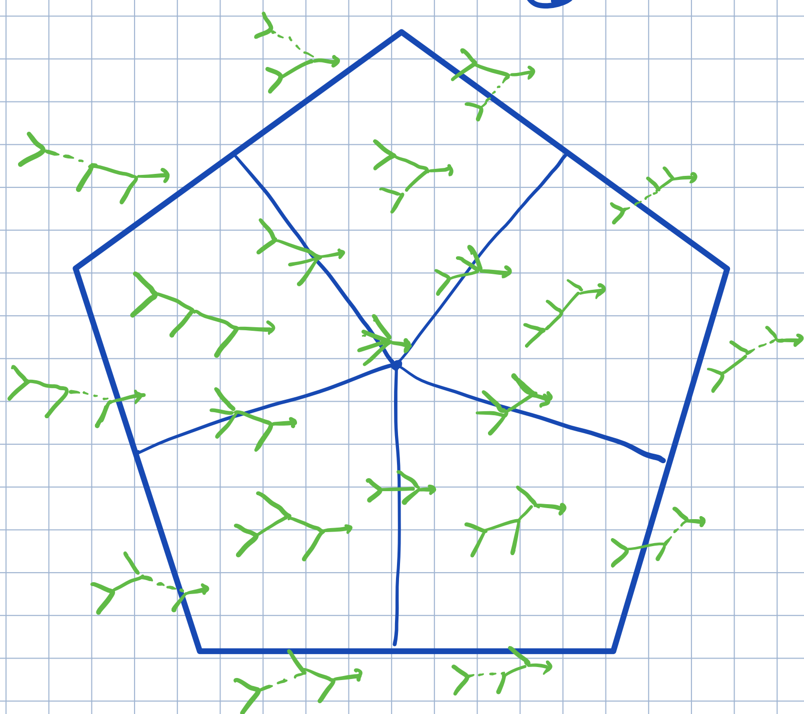
$$\underbrace{\int_{MT_n^\infty} d_{MT} Z}\ +\ \underbrace{\int_{MT_n^\infty} [Q, Z]}_{[d_{V'},\, m_n]} = 0$$

$$\int_{\partial MT_n^\infty} Z = \sum_{T_1, T_2} \int_{MT_1 \times MT_2} Z_{T_2} \circ_i Z_{T_1} = \sum_{\substack{k+l=n+1 \\ 1 \le i \le k \\ k,l \ge 2}} m_k \circ m_l$$



$$Z\Bigg($$  $$\Bigg) + Z\Bigg($$  $$\Bigg) = 0$$

due to $\underline{\underline{m \text{ associative}}}$



<u>dim = 2</u> Gromov - Witten invariants GW classes / A-model

Kontsevich-Manin

Σ Riem. surface of genus g X cpt. Kähler mfd

$X = \mathbb{C}P^N$

<u>small model</u>

$$Z \in \Omega^{\bullet}(\overline{\mathcal{M}}_{g,n}) \otimes \mathrm{Hom}\big(\underbrace{H(X)}_{Q=0}{}^{\otimes n}, \mathbb{R}\big)$$

mod. space of curves

closed form on $\overline{\mathcal{M}}_{g,n}$ $\qquad (d_{geom} + Q) Z = 0$

$$\underbrace{Z(\omega_1, \dots, \omega_n)}_{\text{coh. classes on } X} := \int_{\mathrm{Hol}(\Sigma, X)} \prod_{k=1}^{n} ev_k^* \omega_k$$

holom. maps

$\underline{g=0}$: $\Sigma = \mathbb{C}P^1$ $\qquad ev_k : \mathrm{Hol}(\Sigma, X) \times \mathrm{Conf}_n(\Sigma) \longrightarrow X$

$$\sim \int_{\mathrm{Hol}(\Sigma, X)} \prod_{k=1}^{n} ev_k^* \omega_k \in \Omega^{\bullet}(\mathrm{Conf}_n(\Sigma)) \leftarrow PSL_2(\mathbb{C})\text{-basic}$$

$$\downarrow$$

$n \geq 3$ $\qquad \Omega^{\bullet}\big(\underbrace{\mathrm{Conf}_n(\Sigma)/PSL_2(\mathbb{C})}_{\mathcal{M}_{0,n}}\big)$

$$\mathrm{Hol}(\Sigma, X) \to \overline{\mathrm{Hol}(\Sigma, X)} \xrightarrow{ev_k} X$$

$$\downarrow$$

$$\overline{\mathcal{M}}_{g,n}$$

$\overline{\mathcal{M}_{g,n}(X)}$ Kontsevich's moduli space of stable maps

if $\mathbb{C}P^N$

$$\mathrm{Hol} = \coprod \mathrm{Hol}_d$$

$$\mathrm{Hol} = \coprod \mathrm{Hol}_\beta$$

$$\beta \in H_2(X, \mathbb{Z}) / \text{torsion?}$$

- Enumerative meaning of Z

$$\Big(\int_{\mathcal{M}_{g,n}} Z\Big)(\underbrace{\delta_{C_1}, \dots, \delta_{C_n}}_{\text{cycles on } X}) = \#\{\text{curves of genus } g \text{ in } X \text{ passing through cycles } C_1 - C_n\}$$

amplitude

WDVV eq.