# Gromov-Witten invariants, WDVV, A model

**ref:** Kontsevich, Manin "Gromov-Witten classes, quantum cohomology and enumerative geometry", '94

source $\Sigma$ - Riem surface, genus $g$ , target $X$ cpt, Kähler (e.g. $X = \mathbb{C}P^N$)

$$Z \in \Omega^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathrm{Hom}(H^\bullet(X)^{\otimes n}, \mathbb{C})$$ - closed form on $\mathcal{M}_{g,n}$

moduli space of curves, DM compactification

K-M terminology:

CohFT : $Z^X_{g,n,\beta} : H^\bullet(X,\mathbb{Q})^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$

class in $H^2(X,\mathbb{Z})$

## Construction

for $\omega_1,\dots,\omega_n \in H^\bullet(X)$ , $$Z(\omega_1,\dots,\omega_n) := \int_{\mathrm{Hol}_\beta(\Sigma,X)} \prod_{k=1}^n ev_k^*(\omega_k)$$

**case $g=0$**

$\Sigma = \mathbb{C}P^1$

$\mathrm{Hol}(\Sigma,X) \times \mathrm{Conf}_n(\Sigma) \xrightarrow{ev_k} X$

$$\int_{\mathrm{Hol}(\Sigma,X)} \prod_{k=1}^n ev_k^*\omega_k \in \Omega^\bullet(\mathrm{Conf}_n(\Sigma)) \cup \Omega^\bullet(\mathrm{Conf}_n(\Sigma))^{PSL_2(\mathbb{C})} = \Omega^\bullet(\mathcal{M}_{0,n})$$

**General $g$**

"moduli space of stable maps"

$$\begin{array}{ccccc} \mathrm{Hol}_\beta(\Sigma) & \to & \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{ev_k} & X \\ & & \downarrow & & \\ & & \overline{\mathcal{M}}_{g,n} & & \end{array}$$

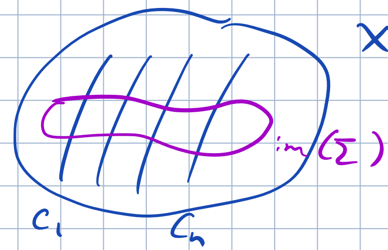
## Enumerative meaning of $Z$:

$$\int_{\mathcal{M}_{g,n}} Z_\beta(\delta_{c_1}, \dots, \delta_{c_n}) = \# \left\{ \begin{array}{l} \text{curves of genus } g \text{ in } X \text{ (in class } \beta) \\ \text{passing through cycles } c_1, \dots, c_n \end{array} \right\}$$

$\delta_{c_1}, \dots, \delta_{c_n}$: Poincaré duals of $c_1, \dots, c_n$ - cycles on $X$

$=:$ Gromov-Witten invariant

$GW_{g,n,\beta}(c_1, \dots, c_n)$

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Ex: $g=0$, $X = \mathbb{C}P^N$, then (thinking $\mathbb{C}P^N = \mathbb{C}^{N+1} - 0 / \mathbb{C}^*$)

$$\mathrm{Hol}_d(\mathbb{C}P^1, \mathbb{C}P^N) = \{ A_p(z, \tilde{z}), \ 0 \le p \le N \} \Big/ \{A_p\} \sim \{c A_p\}, \ c \in \mathbb{C}^*$$

$d$: degree

$A_p$: homog. poly of degree $d$; $[z : \tilde{z}]$ - homog. coords on $\mathbb{C}P^1$

$$= \mathbb{C}P^{\boxed{(d+1)(N+1)-1}} - \mathcal{D}$$

$\mathcal{D}$: divisor where all $A_p$'s have a common root (corresp. "map" is Drinfeld's quasimap)

- $\mathrm{Hol}_{d=0}(\Sigma, X) = \{\text{constant maps}\} \simeq X$

- "quantum cohomology ring" $= H^\bullet(X)$, $m$ ← deformed cup-product

$$\langle m(\omega_1 \otimes \omega_2), \omega_3 \rangle := \sum_d q^d \, GW_{0,3,d}(\omega_1, \omega_2, \omega_3)$$

$\langle \cdot , \cdot \rangle$: Poincaré duality

$q$: "Novikov's parameter"

note: $G_{0,3,0}(\omega_1, \omega_2, \omega_3) = \int_X \omega_1 \wedge \omega_2 \wedge \omega_3$

Ex: $X = \mathbb{C}P^1$, $\mathrm{Hol}_d = \mathbb{C}P^{2d+1} - \mathcal{D}$

$$GW_{0,3,d}(\omega_1, \omega_2, \omega_3) = \int_{\mathbb{C}P^{2d+1}} ev_1^*(\omega_1)\, ev_2^*(\omega_2)\, ev_3^*(\omega_3)$$

$\sim$ want $\overbrace{|\omega_1| + |\omega_2| + |\omega_3|}^{\text{form degree}} = 2 \cdot \underbrace{(2d+1)}_{\dim_{\mathbb{C}} \mathrm{Hol}_d}$

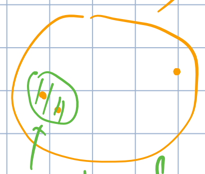
$H^{\bullet}(\mathbb{CP}^1) = \mathrm{span}\{1, [\omega]\}$

Fubini-Study 2-form

$d=0$: $\quad GW_{0,3,0}(1,1,\omega) = 1$

$d=1$: $\quad GW_{0,3,1}(\omega,\omega,\omega) = 1$

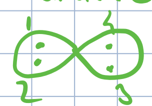
So:
$$1 * 1 = 1$$
$$1 * \omega = \omega$$
$$\omega * \omega = q\, 1$$
$$\omega*\omega*\omega = q\,\omega$$
$$\omega*\omega*\omega*\omega = q^2 \cdot 1$$
$$\omega*\omega*\omega*\omega*\omega = q^2\omega$$
$$\omega*\omega*\omega*\omega*\omega*\omega = q^3$$
$$\vdots$$



contribution of "small instantons" when two points come close

$$\int \qquad \int_{Hol(\Sigma,X)} ev_1^*(\omega)\, ev_2^*(\omega)\, ev_3^*(\omega)\, ev_4^*(\omega)$$

Stratum $\subset \mathcal{M}_{0,4}$



- not a "full" GW invariant

## Gromov-Witten potential

- gen. fun. for GW invariants

Let $\alpha_1, \dots, \alpha_s$ - basis for $H^{\bullet}(X,\mathbb{Z})$

$$\Phi(\underbrace{t_1,\dots,t_s}_{\text{gen. parameters}}) = \sum_{n_1,\dots,n_s\geq 0} \frac{t_1^{n_1}\cdots t_s^{n_s}}{n_1!\cdots n_s!}\, q^d\, GW_{0,\sum n_i, d}(\underbrace{\alpha_1,\dots,\alpha_1}_{n_1},\dots,\underbrace{\alpha_s,\dots,\alpha_s}_{n_s})$$

$$\Phi \in \mathrm{Sym}^{\bullet}\left(H^{\bullet}(X,\mathbb{R})\right)^* [[q]]$$

"big" quantum product $\quad m_\alpha : H(X)\otimes H(X) \longrightarrow H(X)$

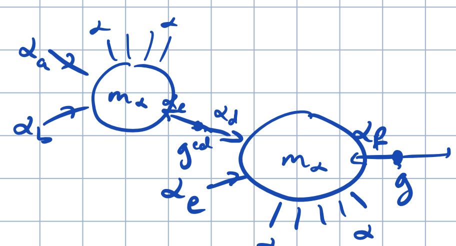
$\alpha \in H^{\bullet}(X)$
"$\sum t_i \alpha_i$"

$$\langle m_\alpha(\omega_1,\omega_2),\omega_3\rangle_{\text{Poincaré}} = \sum_{d\geq 0} \frac{1}{n!} q^d\, GW_{0,n+3,d}(\omega_1,\omega_2,\omega_3,\underbrace{\alpha,\dots,\alpha}_{n})$$

## WDVV equation:

$m_\alpha$ is associative. for each $\alpha \in H^{\bullet}(X)$

equivalently:
$$\frac{\partial^3\Phi}{\partial t_a\,\partial t_b\,\partial t_c}\, g^{cd}\, \frac{\partial^3\Phi}{\partial t_d\,\partial t_e\,\partial t_f} = (a \rightleftarrows e)$$

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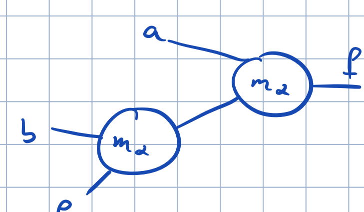

Rem

- $Y = H^\bullet(X)$ is a "Frobenius manifold" (Dubrovin's formalism):
  - equipped with affine flat structure
  - equipped with Riem. metric $g$
  - each tangent space $T_\alpha Y$ is equipped with a comm. assoc. product $m_\alpha$ compatible with $g$ ( $g(x*y,z) = g(x,y*z)$ )
  - equipped with a potential $\Phi \in C^\infty(Y)$ s.t. $g(X*Y, Z) = X \circ Y \circ Z \circ \Phi$ for $X, Y, Z$ flat v.fields

Ex: $X = \mathbb{C}P^1$

$GW_{0,3,d}$ - already know

$$GW_{0,\underset{\geq 3}{n},d}(\underbrace{\overset{\uparrow}{\underset{FS}{\omega}},\dots,\omega}_{k}, \underbrace{1,\dots,1}_{\ell}) = \begin{cases} 0 & \text{if } \ell > 0 \\ 1 & \text{if } \ell = 0, \end{cases}$$

because $\int_{Hol}$ is a basic form on $\mathcal{M}_{0,n}$ $\downarrow$ $\mathcal{M}_{0,k}$, so $\int_{\mathcal{M}_{0,n}} \dots = 0$

$$\underbrace{2n}_{\deg \omega \cdots \omega} = \underbrace{2(2d+1)}_{\dim_{\mathbb{C}} Hol} + \underbrace{2(n-3)}_{\dim \mathcal{M}_{0,n}} \iff \boxed{d=1}$$

Thus:

$$\Phi(\underset{\substack{\uparrow \\ \text{gen. param} \\ \text{for } 1}}{t_0}, \underset{\substack{\uparrow \\ \text{—"—} \\ \text{for } \omega}}{t_1}) = \frac{t_0^2 t_1}{2} + \sum_{n\geq 3} q \frac{t_1^n}{n!}$$

$$= \frac{t_0^2 t_1}{2} + q\left(e^{t_1} - 1 - t_1 - \frac{t_1^2}{2}\right)$$

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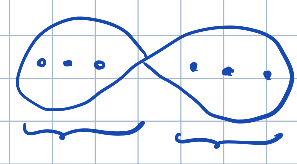
## WDVV - explanation

(1) Keel's relations for $H_\bullet(\overline{\mathcal{M}_{0,n}})$:

Thm (Keel)

$H_\bullet(\overline{\mathcal{M}_{0,n}})$ is generated by $\underset{\substack{\uparrow \\ \text{subset of} \\ \{1,\dots,n\}}}{D_S} = \Big[$  $\Big]$ with $|S|, |S^c| \geq 2$

$S$ $\qquad$ $S^c = \{1\dots n\} - S$

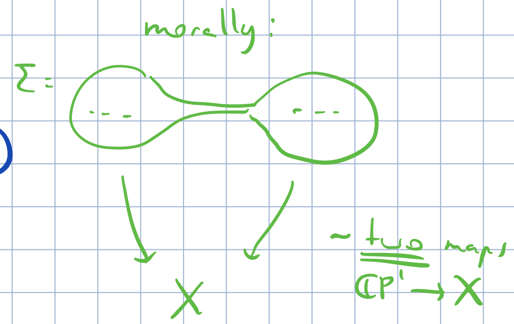
modulo relations

- $D_S = D_{S^c}$
- for $\{i,j,k,\ell\}$ distinct, $$\sum_{\substack{i,j \in S \\ k,\ell \in S^c}} D_S = \sum_{\substack{i,k \in S \\ j,\ell \in S^c}} D_S = \sum_{\substack{i,\ell \in S \\ j,k \in S^c}} D_S$$
- $D^S \cap D^T = 0$ unless $S \subset T$ or $T \subset S$ or $S^c \subset T^c$ or $T^c \subset S^c$

(2) factorization property of GW classes

$$Z(\omega_1,\dots,\omega_n) = \int_{\mathcal{H}ol(\Sigma,X)} ev_1^*\omega_1 \cdots ev_n^*\omega_n \in \Omega^{\cdot}(\mathcal{M}_{0,n})$$

$$Z(\omega_1\dots\omega_n)\big|_{D_S} = \sum_{i,j} Z(\omega_S,\alpha_i)\, g^{ij}\, Z(\omega_{S^c},\alpha_j)$$




$\Rightarrow$

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{a,b\in S\\ e,f\in S^c}}\int_{D_S} Z(\alpha_a,\alpha_b,\alpha_e,\alpha_f,\underbrace{\overset{1}{\alpha},\dots,\overset{n}{\alpha}}_{n}) \overset{\text{factorization}}{=}$$

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{a,b\in S\\ e,f\in S^c}}\int_{\mathcal{M}_{S,c}} Z(\alpha_a,\alpha_b,\overbrace{\alpha,\dots,\alpha}^{S},\alpha_c)\, g^{cd}\int_{\mathcal{M}_{S^c,d}} Z(\alpha_e,\alpha_f,\overbrace{\alpha\dots\alpha}^{S^c},\alpha_d)$$

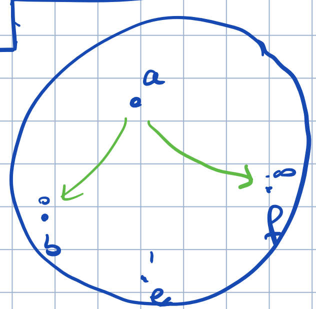
$$=\sum_{k+l=n} GW_{0,k+3}(\alpha_a,\alpha_b,\alpha_c,\underbrace{\alpha\dots\alpha}_{k})\, g^{cd}\, GW_{0,l+3}(\alpha_d,\alpha_e,\alpha_f,\underbrace{\alpha\dots\alpha}_{l})$$

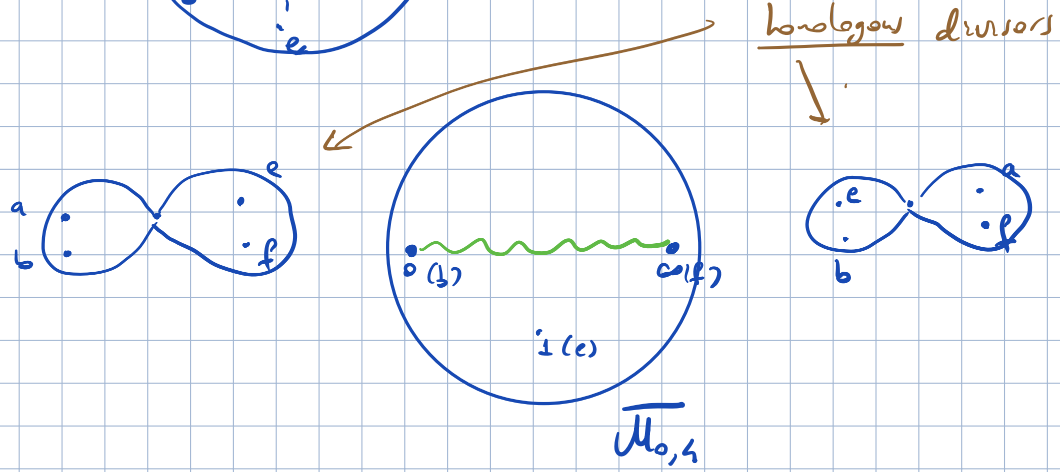
$$=\frac{\partial^3\Phi}{\partial t_a\,\partial t_b\,\partial t_c}\, g^{cd}\,\frac{\partial^3\Phi}{\partial t_d\,\partial t_e\,\partial t_f}$$

$\|$ $\nwarrow$ rel

$(a \rightleftarrows e)$

Ex: $(\alpha = 0)$:



$\underline{Ex}$: $\Sigma = \mathbb{CP}^1$, $\underline{\underline{X = \mathbb{CP}^2}}$ (Kontsevich-Manin)

$t_0 \quad t_1 \quad t_2$ - gen. parameters

$$H^\bullet(X) = \mathrm{span}(1, \omega, \omega^2)$$

$$GW_{0,n,d}(\underbrace{1,\dots,1}_{n_0}, \underbrace{\omega,\dots,\omega}_{n_1}, \underbrace{\omega^2,\dots,\omega^2}_{n_2}) = \begin{cases} 0 & \text{if } n_0 > 0 \\ 0 & \text{if } n_2 \neq 3d-1 \\ N(d)\cdot d^{n_1} \quad (*) & \text{if } n_0 = 0,\ n_2 = 3d-1 \end{cases}$$

$n \geq 4$, $n_0 + n_1 + n_2 = n$ — balancing condition

balancing condition:

$$\underbrace{2n_1 + 4n_2}_{\text{form degree}} = 2(\underbrace{3(d+1)-1}_{\dim_{\mathbb{C}} \mathrm{Hol}} + \underbrace{n-3}_{\dim \mathcal{M}_{0,n}}) = 2(3d-1+n_1+n_2) \Leftrightarrow n_2 = 3d-1$$

$$\boxed{\text{Here } N(d) = \#\text{ rational curves of degree } d \text{ in } \mathbb{CP}^2 \text{ passing through } 3d-1 \text{ points}}$$

$N(1) = 1$ line through $3\cdot 1 - 1 = 2$ points in $\mathbb{CP}^2$

$N(2) = 1$ conic through $3\cdot 2 - 1 = 5$ points in $\mathbb{CP}^2$

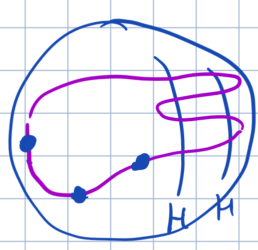
$N(3) = 12$ rational cubic through 8 points

$N(4) = 620$

$N(5) = 87304$

Factor $d^{n_1}$ in $(*)$ - since a curve of deg $= d$ intersects a hyperplane $H \subset \mathbb{CP}^2$ in $d$ points

$H \updownarrow$ Poincaré dual $\omega$




Or: because $(ev_i^* \omega)_{(2,0)} = d\cdot\omega$

deg. conf. along hol.

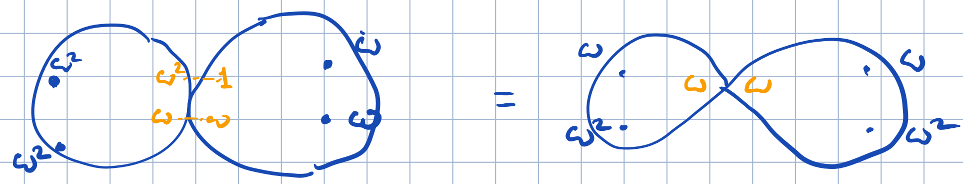
by a hol. map of deg $= d$

So, GW potential:

$$\Phi(t_0,t_1,t_2) = \frac{t_0^2 t_2}{2} + \frac{t_0 t_1^2}{2} + \underbrace{\sum_{d\geq 1} \frac{N(d)}{(3d-1)!} q^d t_2^{3d-1} e^{dt_1}}_{\varphi(t_1,t_2;q)}$$

• all $N(d)$'s are fixed by WDVV + "init. cond." $N(1) = 1$.

Nontrivial component of WDVV:



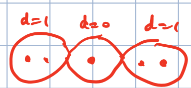

$$\text{WDVV} \iff \varphi_{t_2t_2t_2} = (\varphi_{t_1t_1t_2})^2 - \varphi_{t_1t_1t_1}\varphi_{t_1t_2t_2}$$

$$\iff \mathcal{N}(d) = \sum_{k+l=d} \mathcal{N}(k)\mathcal{N}(l)\, k^2 l \left( l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right) \quad , d \geq 2$$

<u>Ex:</u> $\Sigma = \mathbb{C}P^1$ , $X = \mathbb{C}P^1$

$$Z(\underbrace{\omega, \dots, \omega}_{5}) = q^2 + q \cdot \begin{pmatrix} \text{vol. 4-form} \\ \text{on } \mathcal{M}_{0,5} \end{pmatrix}$$

ck $\omega * \omega * \omega * \omega = q^2 1$ GW-invariant




$$Z(\underbrace{\omega, \dots, \omega}_{7}) = q^3 + q^2 \begin{pmatrix} \text{a closed} \\ \text{4-form on} \\ \mathcal{M}_{0,7} \end{pmatrix} + q \begin{pmatrix} \text{vol 8-form} \\ \text{on } \mathcal{M}_{0,7} \end{pmatrix}$$



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## Path integral formalism

(genus 0)

$$Z(\omega_1, \dots, \omega_n) = \langle \tilde{\mathcal{O}}_{\omega_1}(z_1) \dots \tilde{\mathcal{O}}_{\omega_n}(z_n) \rangle$$

$$= \int_{\text{Fields}} e^{-\frac{1}{\hbar} S_A} \quad \tilde{\mathcal{O}}_{\omega_1}(z_1) \dots \tilde{\mathcal{O}}_{\omega_n}(z_n)$$

"extended" observables $\mathcal{O}^{(0)}_\omega + \mathcal{O}^{(1)}_\omega + \mathcal{O}^{(2)}_\omega$

$\underbrace{\varphi : \Sigma \to X}_{\text{bosonic}}$ ; $\chi \in \varphi^*(TX)$ $gh=+1$

$\psi \in \Omega^{1,0}(\Sigma, \varphi^*(T^{0,1}X))$ $gh=-1$

$\bar{\psi} \in \Omega^{0,1}(\Sigma, \varphi^*(T^{1,0}X))$ $gh=-1$

fermionic

$$S_A = \int_\Sigma \left( \frac{1}{2} g_{IJ} \partial\varphi^I \bar{\partial}\varphi^J + i\psi^{\bar{i}} \bar{D}\chi^i g_{i\bar{i}} + i\bar{\psi}^i D\chi^{\bar{i}} g_{i\bar{i}} - R_{i\bar{i}j\bar{j}} \bar{\psi}^i \psi^{\bar{i}} \chi^j \chi^{\bar{j}} \right)$$

$$\tilde{\mathcal{O}}_\omega \qquad \mathrm{Map}(\Sigma, X) \times \Sigma \xrightarrow{ev} X$$

$$\tilde{\mathcal{O}}_\omega \xleftarrow{ev^*} \omega \text{ - } p\text{-form}$$

$$\omega = \omega_{I_1 \dots I_p}(x)\, dx^{I_1} \dots dx^{I_p} \rightsquigarrow \tilde{\mathcal{O}}_\omega = \omega_{I_1 \dots I_p}(x)\,(\chi^{I_1} + d\varphi^{I_1}) \dots (\chi^{I_p} + d\varphi^{I_p})$$

- Another viewpoint

$$\int \underbrace{e^{\cdots}}_{\substack{\text{MQ representative} \\ \text{for } \delta_{\mathrm{Hol} \subset \mathrm{Fields}}}} \tilde{\mathcal{O}}_{\omega_1} \dots \tilde{\mathcal{O}}_{\omega_n} = \int_{\mathrm{Hol}} \tilde{\mathcal{O}}_{\omega_1} \dots \tilde{\mathcal{O}}_{\omega_n}$$

$$\begin{array}{c} \bigcirc \leftarrow \underset{\bar{\rho},\,\bar{\psi}}{\Omega^{0,1}(\Sigma, T^{1,0}X)} \oplus \underset{\rho,\,\psi}{\Omega^{1,0}(\Sigma, T^{0,1}X)} \\ \downarrow \\ \mathrm{Map}(\Sigma, X) \end{array}$$

bundle of holomorphicity equations

**Mathai-Quillen:**

$$\begin{array}{c} E \\ s \text{ (section)} \uparrow\downarrow \\ M \end{array} \quad \text{vector bundle}$$

want $\delta_{s^{-1}(0)}$ the Euler class

$$= \frac{1}{\sqrt{\det g_{fib}}} e^{-g_{fib}(s,s)} \nabla_M s \rightsquigarrow \underset{\substack{\uparrow \\ \text{pushforward}}}{\int dp}\, e^{i\langle p, s\rangle - \overset{\substack{\text{smearing parameter} \\ \downarrow}}{\varepsilon} g_{fib}^{-1}(p,p)} \nabla_M s \longrightarrow$$

$$\begin{array}{c} E^* \\ \downarrow \\ M \end{array}$$

$$\longrightarrow \int dp\, D\pi\, e^{i\langle p, s\rangle + i\langle \pi, \nabla_M s\rangle - \varepsilon g_{fib}(p,p)}$$

pushforward

$$\begin{array}{c} E^* \oplus E^*[-1] \\ \downarrow \\ M \end{array}$$

$$\rightsquigarrow \int dp\, D\pi\, e^{i\langle p,s\rangle + i\langle \pi, \nabla_M s\rangle - \varepsilon(p,p) + \varepsilon(\pi, \overset{\substack{\text{curvature} \\ \downarrow}}{F_M} \pi)}$$

(cf. (11.12), p.108 in Cordes-Moore-Ramgoolam)

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Ex:

$$\begin{array}{c} \mathcal{O}_M \\ \downarrow \\ M \end{array} \; \big) \; s\text{-function}$$

$$\delta^\varepsilon_{s^{-1}(0)} = \frac{1}{\sqrt{\pi\varepsilon}} e^{-\frac{s^2}{2\varepsilon}} d_M s = \frac{1}{\pi} \int dp\, e^{ip\, s(x) - \varepsilon \frac{p^2}{2}} d_M s$$

$$= \frac{1}{i\pi} \int dp\, d\pi\, e^{ip\, s(x) + i\pi\, d_M s - \frac{\varepsilon p^2}{2}}$$