Felix Klein seminar
4/11/2024

A gluing formula for heat kernels

w/ Konstantin Wernli, arXiv: 2404.00156

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Plan

- Green's fcn $\xleftrightarrow[\text{Laplace}]{}$ heat kernel
- DN operator, $DN - \sqrt{\Delta_g + m^2}$
- gluing formula for Green's functions
- gluing for heat kernels I
- gluing for heat kernels II
- Examples → Ex of DN operators: cylinder, disk
- Path integral interpretation [heuristic]
- Graph setting

Felix Klein seminar
Thu 4/11/2024 2-3pm
125 HH

A gluing formula for heat kernels

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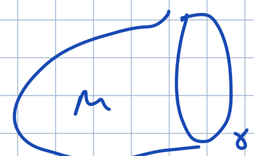
Motivation:
Feynman diagrams in QFT
$\int_{M^{\times n}} G \cdots G$ - diverge
$G = \int_0^\infty dt\, K(t)$
regularization
$\int_\epsilon^\infty dt\, K(t)$ OR $\int_0^\infty dt\, t^\epsilon K(t)$



Let M - Riemannian mfd with bdry γ.

- $G^{M,\gamma}_{m^2}(x,y)$ - Green's fun. for $\Delta + m^2$ on M with Dirichlet b.c. on γ
  (Δ: d^*d - non-negative Laplace-Beltrami)
- $K^{M,\gamma}(x,y|t)$ - heat kernel (int. kernel of $e^{-t\Delta}$), with Dirichlet b.c. on γ

Note: $G_{m^2}(x,y) = \int_0^\infty dt\, \underbrace{e^{-m^2 t} K(x,y|t)}_{\text{int. kernel of } e^{-t(\Delta+m^2)}} = L(K(x,y|t))$

L: Laplace transform, $f(t) \mapsto \tilde{f}(s=m^2)$

So, $K(x,y|t) = L^{-1}(G_{m^2}(x,y)) = \frac{1}{2\pi i}\int dm^2\, e^{m^2 t}\, G_{m^2}(x,y)$

Ex: $M = \mathbb{R} \rightsquigarrow K(x,y|t) = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \xrightarrow{L} G(x,y) = \frac{e^{-m|x-y|}}{2m}$

$M = [0,\infty) \rightsquigarrow K = \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right) \xrightarrow{L} G(x,y) = \frac{1}{2m}\left(e^{-m|x-y|} - e^{-m(x+y)}\right)$

Extension (or "Poisson") and Dirichlet-to-Neumann operators.

- for $\eta \in C^\infty(\gamma)$, let $\varphi_\eta \in C^\infty(M)$ be the (unique) $(\Delta+m^2)$-harmonic extension of η into M:

$$\begin{cases} \varphi_\eta|_\gamma = \eta \\ (\Delta+m^2)\varphi_\eta = 0 \end{cases}$$

- $E^{\gamma,M}: C^\infty(\gamma) \to C^\infty(M)$, $\eta \mapsto \varphi_\eta$ "extension operator", $E(\eta)(x) = \int_\gamma dy\, \underbrace{\boxed{\partial^n_y G^{M,\gamma}_{m^2}(x,y)}}_{\text{int kernel}}\, \eta(y)$

- $DN^{\gamma,M}: C^\infty(\gamma) \to C^\infty(\gamma)$, $\eta \mapsto \partial^n \varphi_\eta$ "Dirichlet-to-Neumann operator",
$DN(\eta)(x) = \int_\gamma dy\, \underbrace{\boxed{\partial^n_x \partial^n_y G(x,y)}}_{\text{int. kernel}}\, \eta(y)$

- DN vs $\sqrt{\Delta_\gamma + m^2}$

Consider the operator $\mathscr{K} := \sqrt{\Delta_\gamma + m^2} : C^\infty(\gamma) \to C^\infty(\gamma)$ - PDO of order +1

$\mathscr{K} = DN^{\gamma, \gamma\times[0,\infty)}$

if $\Delta_\gamma \eta = \lambda \eta$, then harm. extension to $\gamma \times [0,\infty)$ is: $\eta \cdot e^{-\sqrt{\lambda+m^2}\, t}$
$\Rightarrow DN: \eta \to \sqrt{\lambda+m^2}\,\eta$

$DN' := DN - \mathscr{X}$ is:

- a <u>smoothing</u> operator if M has product metric near γ
- PDO of order ≤ 0 $\xrightarrow{\text{if } \gamma \text{ cpt.}}$ DN' is bounded.
- if $\dim M = 2$, PDO of order ≤ -2.

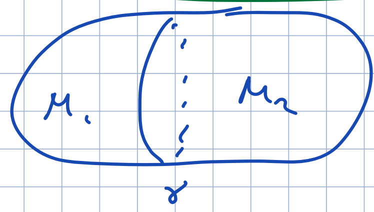
Ex: $M = \gamma \times [0, L]$ cylinder

$\Delta_\gamma \eta_k = \omega_k \eta_k$

$\rightsquigarrow DN(\eta_k) = \underbrace{\sqrt{m^2+\omega_k}\, \coth L\sqrt{m^2+\omega_k}}_{\lambda_k} \cdot \eta_k$

$\lambda_k - \sqrt{m^2+\omega_k}$ - eigenvalue of DN'

- <u>Interface DN operator</u>: if $M = M_1 \cup_\gamma M_2$
(or "total")



$$\mathbb{D}^{\gamma,M} := DN^{\gamma,M_1} + DN^{\gamma,M_2} : C^\infty(\gamma) \to C^\infty(\gamma)$$

fact: $\underbrace{(\mathbb{D}^{\gamma,M})^{-1}(x,y)}_{\text{int. kernel}} = G^M(x,y)$ - restriction of G^M to $\gamma \times \gamma$.

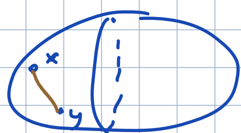
Carron, '02

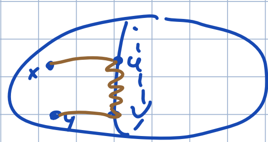
<u>Gluing f-la for Green's functions</u>

[ref: Kandel-Mnev-Wernli, '21]

Let $M = M_1 \cup_\gamma M_2$. Then for $x \in M_a$, $y \in M_b$, $a,b \in \{1,2\}$:

$$(*) \quad G^M(x,y) = \delta_{ab} G^{M_a}(x,y) + \int_{\gamma\times\gamma} du\, dv\ \partial_u^n G^{M_a}(x,u)\, \mathbb{D}^{-1}(u,v)\, \partial_v^n G^{M_b}(v,y)$$





<u>Gluing f-la I for heat kernels</u>

Applying L^{-1} to $(*)$: (product $\xrightarrow{L^{-1}}$ convolution in t)

$$(\mathrm{I}) \quad K^M(x,y|t) = \delta_{ab} K^{M_a}(x,y|t) + \int_{\substack{t_0+t_1+t_2=1 \\ t_i>0}} dt_0\, dt_1 \int_{\gamma\times\gamma} du\, dv\ \partial_u^n K^{M_a}(x,u|t_0) \boxed{L^{-1}(\mathbb{D}^{-1})(u,v|t_1)}\, \partial_v^n K^{M_b}(v,y|t_2)$$

Ex: $M = \mathbb{R} = \underset{M_1}{(-\infty,0]} \underset{\{0\}}{\cup} \underset{M_2}{[0,+\infty)}$

$x,y>0$

$$K^M(x,y|t) - K^{M_2}(x,y|t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{(x+y)^2}{4t}} \underset{(\mathrm{I})}{=} \int_{\substack{t_0+t_1+t_2=1 \\ t_i>0}} dt_0\, dt_1\ \frac{-xe^{-\frac{x}{4t_0}}}{\sqrt{4\pi t_0^3}}\ \underset{\substack{\uparrow \\ L^{-1}(\mathbb{D}^{-1}) \\ = L^{-1}(\frac{1}{2m})}}{\frac{1}{\sqrt{2\pi t_1}}}\ \frac{-ye^{-\frac{y}{4t_2}}}{\sqrt{4\pi t_2^3}}$$

<u>Gluing f-la II for heat kernels</u>

Want to express $L^{-1}(\mathbb{D}^{-1})$ in (I) via $K^{M_a}, K^{M_b}, K^\gamma$!

$$\mathbb{D} = DN^{\gamma,M_1} + DN^{\gamma,M_2} = \underbrace{2\sqrt{\Delta_\gamma + m^2}}_{A} + \underbrace{DN'^{\gamma,M_1} + DN'^{\gamma,M_2}}_{\text{"small"}} = A + \mathbb{D}'$$

(#) $\mathbb{D}^{-1} = A^{-1} - A^{-1}\mathbb{D}' A^{-1} + A^{-1}\mathbb{D}'A^{-1}\mathbb{D}'A^{-1} - \dots$ ← converges for $\mathrm{Re}\, m^2 > C$ under an assumption on M_1, M_2:

- operator $\mathbb{D}'$ is bounded
- $\forall \delta>0\ \exists C \gg 1$ s.t. $\forall\, \mathrm{Re}\, m^2 > C$ one has $\|\mathbb{D}'\| < \delta |m|$

assumption holds e.g. for product metric near γ.

note:

- $A^{-1} = \left(\mathbb{D}^{\gamma,\gamma\times\mathbb{R}}\right)^{-1} = G^{\gamma\times\mathbb{R}}\Big|_{(\gamma\times\{0\})\times(\gamma\times\{0\})}$

$\Rightarrow L^{-1}(A^{-1})(u,v|t) = K^{\gamma\times\mathbb{R}}((u,0),(v,0)|t)$

$\qquad \| \leftarrow K^{M\times N}_{(t)} = K^M_{(t)} \otimes K^N_{(t)}$

$$K^\gamma(u,v|t)\, K^{\mathbb{R}}(0,0|t) = \frac{1}{\sqrt{4\pi t}} K^\gamma(u,v|t)$$

use factorization of K

- $L^{-1}(\underbrace{DN^{\prime\,\gamma,M_a}}_{DN - \sqrt{\Delta+m^2}})(u,v|t) = \partial^n_u \partial^n_v \left(K^{M_a}(u,v|t) - K^{\gamma\times\mathbb{R}_+}(u,v|t)\right)$

$\qquad = \partial^n_u\partial^n_v K^{M_a}(u,v|t) - \frac{1}{\sqrt{4\pi t^3}} K^\gamma(u,v|t)$

$$\Rightarrow L^{-1}(\mathbb{D}') = \sum_{a=1}^{2} \partial^n_u \partial^n_v K^{M_a}(u,v|t) - \frac{1}{\sqrt{\pi t^3}} K^\gamma(u,v|t) =: K'(u,v|t)$$

substituting $L^{-1}(\#)$ into (I), we get:

$$\text{(II)}\quad K^M(x,y|t) = \delta_{ab} K^{M_a}(x,y|t) + \sum_{k\geq 0} (-1)^k \int_{\substack{\sum_{i=0}^{2k+2} t_i = t,\\ t_i>0}} \prod_{i=0}^{2k+1} dt_i \int_{\gamma^{2k+2}} \prod_{i=0}^{2k+1} du_i$$

$$\partial^n_{u_0} K(x,u_0|t_0) \left(\prod_{i=0}^{k-1} \frac{1}{\sqrt{4\pi t_{2i+1}}} K^\gamma(u_{2i},u_{2i+1}|t_{2i+1})\, K'(u_{2i+1},u_{2i+2}|t_{2i+2})\right) \cdot \frac{1}{\sqrt{4\pi t_{2k+1}}} K^\gamma(u_{2k},u_{2k+1}|t_{2k+1}) \cdot$$

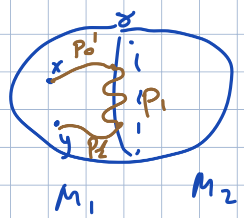
$$\cdot\, \partial^n_{u_{2k+1}} K^{M_b}(u_{2k+1},y|t_{2k+2})$$

Path integral (heuristic) interpretation

Feynman-Kac f-la $K^M(x,y|t) = \int_{P^t_M(x,y)} Dp\; e^{-S(p)}$, $S(p) = \int_0^t \left(\frac{\overset{g_{p(\tau)}(\dot p(\tau),\dot p(\tau))}{\dot p^2}}{4} - \frac{\overset{\text{scalar curvature}}{R(p(\tau))}}{6} \right) d\tau$

$P^t_M(x,y) = \{\text{paths } p:[0,t]\to M \mid p(0)=x,\ p(t)=y\}$

Now let $M = M_1 \cup_\gamma M_2$. $P^t_M(x,y) = \underbrace{P^t_{M_1\setminus\gamma}(x,y)}_{\text{paths not crossing }\gamma} \amalg \underbrace{P^t_{M,\gamma}(x,y)}_{\text{paths crossing }\gamma}$



paths not touching γ except at end point

$$P^t_{M,\gamma}(x,y) = \coprod_{\substack{t_0+t_1+t_2=t\\ t_i>0}} \coprod_{u,v\in\gamma} P'^{t_0}_{M_1,\gamma}(x,u) \times P^{t_1}_M(u,v) \times P'^{t_2}_{M_1,\gamma}(v,y)$$

$$\int_{\cdots} e^{-S} \rightsquigarrow \quad \underbrace{\int_{P^t_M}}_{K^M} = \underbrace{\int_{P^t_{M_1\setminus\gamma}}}_{K^{M_1,\gamma}} + \int_{\Delta^2} dt_0\, dt_1 \int_{\gamma^2} du\, dv\; \underbrace{\int_{P'^{t_0}_{M_1,\gamma}} Dp_0'\, e^{-S(p_0')}}_{\overset{\text{WANT}}{=} \partial^n_u K^{M_1,\gamma}(x,u)} \underbrace{\int_{P^{t_1}_M} Dp_1\, e^{-S(p_1)}}_{K^M(u,v)} \underbrace{\int_{P'^{t_2}_{M_1,\gamma}} Dp_2\, e^{-S(p_2')}}_{\overset{\text{WANT}}{=} \partial^n_v K^{M_1,\gamma}(v,y)}$$

Graph setting X graph (finite, simple), $(\Delta^X)_{ij} = \begin{cases} val(i) & \text{if } i=j \\ 1 & \text{if } (i,j)\in E \\ 0 & \text{otherwise} \end{cases}$ - graph Laplacian

$\Delta^X : C^0(X) \to C^0(X)$.

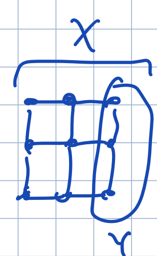
path sum formulas:

$$\overset{\langle y|\frac{1}{\Delta^X+m^2}|x\rangle}{G_{m^2}(x,y)} = \sum_{\substack{p\in P_X(x,y)\\ \| \\ (v_0=x,v_1,\dots,v_n=y)}} \underbrace{\prod_{j=0}^{n} \frac{1}{m^2+val(v_j)}}_{w(p)}$$

$$K(x,y|t) = \langle y|e^{-t\Delta^X}|x\rangle = \sum_{p\in P_X(x,y)} \underbrace{\int_{\substack{t_0+\dots+t_n=t\\ t_i>0}} dt_1\cdots dt_n\; e^{-\sum_{i=0}^n t_i\, val(v_i)}}_{W(p)} \quad (@)$$

Let $Y\subset X$ subgraph ("boundary")

$$\Delta^X + m^2 = \begin{array}{c|cc} & X\setminus Y & Y \\ \hline X\setminus Y & \hat A = \Delta^{X,Y}+m^2 & \hat B \\ Y & \hat C & \hat D \end{array}, \quad G^X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$



$\cancel{\hat C B + \hat D D = 1_Y}$ $\quad D^{-1} = -\hat C \hat A^{-1} \hat B + \hat D$

"Dirichlet problem": $\begin{cases}(\Delta^X+m^2)\varphi(x)=0, & x\in X\setminus Y\\ \varphi(x)=\eta(x), & x\in Y\end{cases}$

"extension operator"

$E^{Y,X}: C^0(Y)\to C^0(X)$

$\eta\mapsto\varphi_\eta=\begin{pmatrix}BD^{-1}\eta\\ \eta\end{pmatrix}$

$G^{X,Y}=\hat{A}^{-1}: C^0(X\setminus Y)\circlearrowleft$, $DN^{Y,X}=D^{-1}: C^0(Y)\circlearrowleft$

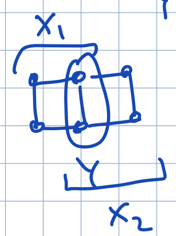
"Dirichlet-to-Neumann operator"

$\boxed{\text{Rem: } BD^{-1}=-\hat{A}^{-1}\hat{B}\sim \text{"}\partial^n G^{X,Y}\text{"}}$ "normal derivative"

same structure as in continuum case

Gluing of Green's fun:

$X=X_1\cup_Y X_2$: $\quad G^{X,Y}=\hat{A}^{-1}\underset{\text{Schur's complement f-la}}{=}A-BD^{-1}C=G^X|_{X\setminus Y}-\underbrace{(BD^{-1})}_{E^{Y,X}}\underbrace{D}_{(\mathbb{D}^{Y,X})^{-1}}\underbrace{(D^{-1}C)}_{(E^{Y,X})^T}$



$\mathbb{D}^{Y,X}=DN^{Y,X_1}+DN^{Y,X_2}-(\Delta^Y+m^2)$

i.e. $G^X(x,y)=G^{X,Y}(x,y)+\sum_{u,v\in Y}E^{Y,X}(x,u)\,\mathbb{D}^{-1}(u,v)\,(E^{Y,X})^T(v,y)$

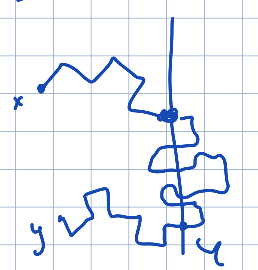
Gluing of heat kernels (I)

apply L^{-1}

$$K^X=K^{X,Y}+\int_{\substack{t_0+t_1+t_2=t\\ t_i>0}}dt_1\,dt_2\;\underbrace{\mathcal{E}^{Y,X}(t_1)}_{L^{-1}(E^{Y,X})=-G^{X,Y}\hat{B}}\,L^{-1}\big((\mathbb{D}^{Y,X})^{-1}\big)(t_0)\,\big(\mathcal{E}^{Y,X}(t_2)\big)^T$$

"normal derivative"

(@) $\sim$ splitting of paths on X by intersections with Y.



On $\mathcal{E}^{Y,X}$:

$\eta\in C^0(Y)\rightsquigarrow\ \underset{\mathcal{E}^{Y,X}(t)\eta}{\overset{}{\varphi(t)}}\in C^0(X)$ s.t. $\begin{cases}(\partial_t+\Delta^X)\varphi=0, & t>0\\ \varphi(0,x)=0, & x\in X\setminus Y\\ \varphi(t,y)=\delta(t), & y\in Y\end{cases}$

Gluing f-la (II):

$\mathbb{D}=\underbrace{\mathrm{Diag}(\mathrm{val}^Y(v)+m^2)}_{\Lambda}-\mathbb{D}'$

$\rightsquigarrow\mathbb{D}^{-1}=\Lambda^{-1}+\Lambda^{-1}\mathbb{D}'\Lambda^{-1}+\ldots$

$$\Delta_X+m^2=\begin{array}{c}X_1\setminus Y\\ X_2\setminus Y\\ Y\end{array}\left(\begin{array}{cc|c}\hat{A}_1:=\Delta_{X_1,Y}+m^2 & 0 & \hat{B}_1\\ 0 & \hat{A}_2 & \hat{B}_2\\ \hline \hat{C}_1 & \hat{C}_2 & \hat{D}\end{array}\right)$$

$\mathbb{D}':=\underset{\text{adjacency matrix}}{I^Y}+\hat{C}_1\hat{A}_1^{-1}\hat{B}_1+\hat{C}_2\hat{A}_2^{-1}\hat{B}_2$

$L^{-1}(\mathbb{D}')(t)=\delta(t)I^Y+\sum_{a=1}^{2}\hat{C}_a K_{X_a,Y}(t)\hat{B}_a=:\mu(t)$

$$\rightsquigarrow \underset{x,y\in X_a}{K^{X}(x,y\,|\,t)} = \delta_{ab} K^{X_a}(x,y\,|\,t) + \sum_{k\geq 0} \int_{\substack{\sum_{i=0}^{2k+2} t_i = t \\ t_i \geq 0}} \prod_{i=1}^{2k+2} dt_i \sum_{u_0,\dots,u_k \in Y} \varepsilon^{Y,X_a}(x,u_0\,|\,t_0)$$

$$\prod_{i=0}^{k-1}\left(e^{-t_{2i+1}\mathrm{val}_X(u_i)} \mu(u_i,u_{i+1}\,|\,t_{2i+2}) \right) \cdot e^{-t_{2k+1}\mathrm{val}_X(u_k)} \cdot \left(\varepsilon^{Y,X_b}\right)^T(u_k,y\,|\,t_{2k+2})$$