talk at
ND departmental orientation
for new grad students
8/6/2020
2:00

Functorial field theory and simple examples

Plan:
- I. Atiyah-Segal axioms of QFT
- II. Example: Dijkgraaf-Witten model
- III. Example: 2d Yang-Mills theory
- IV. Example: classical Chern-Simons theory

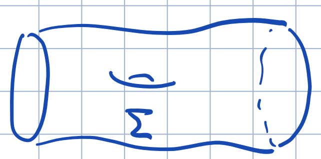
I. Atiyah-Segal's axioms. [outline]

n-dim. QFT is an assignment

DATA

(1) (n−1)-dim. closed manifold γ $\longmapsto$ vector space $/\mathbb{C}$ $\mathcal{H}_\gamma$ "space of states"

(2) n-dim. mfd Σ with boundary $\partial\Sigma = \gamma_{out} \amalg \overline{\gamma_{in}}$ (cobordism) $\longmapsto$ linear map $Z_\Sigma : \mathcal{H}_{\gamma_{in}} \to \mathcal{H}_{\gamma_{out}}$ "partition function"



γ_{in} γ_{out}

Axioms:

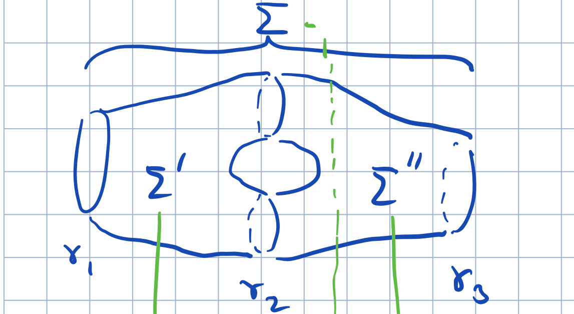
(a) multiplicativity: $\mathcal{H}_{\gamma_1 \amalg \gamma_2} = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$

$Z_{\Sigma_1 \amalg \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$

(b) gluing: if $\gamma_1 \xrightarrow{\Sigma'} \gamma_2$, $\gamma_2 \xrightarrow{\Sigma''} \gamma_3$ two cobordisms,

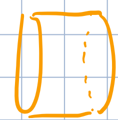
then for the glued cobordism

$$\Sigma = \Sigma' \cup_{\gamma_2} \Sigma'' , \quad Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'}$$



$$\mathcal{H}_{\gamma_1} \xrightarrow{Z_{\Sigma'}} \mathcal{H}_{\gamma_2} \xrightarrow{Z_{\Sigma''}} \mathcal{H}_{\gamma_3}$$

composition

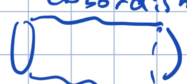


has to be modified in non-topological case

(c) normalization: $\mathcal{H}_\emptyset = \mathbb{C}$, $Z_{\gamma \times [0,1]} = id : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$

So: a QFT is a functor between (symmetric, monoidal) categories

$$\mathrm{Cob}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

	Cob_n	$\mathrm{Vect}_{\mathbb{C}}$
Objects	closed $(n-1)$-mflds 	vector spaces
morphisms	cobordisms 	linear maps
composition $\circ$	gluing 	composition of maps
$\otimes$	$\amalg$	$\otimes$

Rem: One can consider manifolds - <u>without</u> loc. geom. structure (but maybe with orientation) - TQFT (topological) - Atiyah's case

or <u>with</u> loc. geom. structure (e.g. metric) - more general QFT - Segal's case.

Rem For Σ closed n-mfd, $\emptyset \xrightarrow{\Sigma} \emptyset$, $Z_\Sigma \in \mathbb{C}$ a number.

TQFT $\overset{\text{diff.}}{=}$ invariants of "manifolds behaving nicely w.r.t. cutting-gluing.

<u>Ex</u>: {1-dimensional TQFTs} $\longleftrightarrow$ {fin. dim. v. sp. $V = \mathcal{H}_{pt}$ }

then e.g. $Z(S^1) = \dim V$

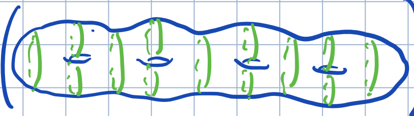
<u>Ex</u>: {2-dimensional TQFTs} $\overset{\text{(Dijkgraaf)}}{\longleftrightarrow}$ { <u>Frobenius algebras</u> $V = \mathcal{H}_{S^1}$

$m = Z(\;\;) : V^{\otimes 2} \to V$ product

$\Delta = Z(\;\;) : V \to V^{\otimes 2}$ coproduct

$1 = Z(\;\;) : \mathbb{C} \to V$ unit

$tr = Z(\;\;) : V \to \mathbb{C}$ counit (trace) }

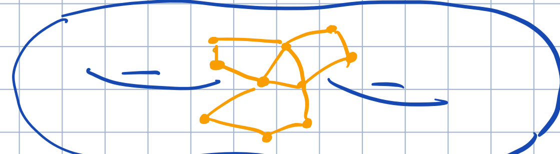
then e.g. $Z\Big($  $\Big) = tr\big((m\Delta)^h \circ 1\big)$

genus h

II. Dijkgraaf - Witten model

(special case)

Σ - closed surface with cell decomposition

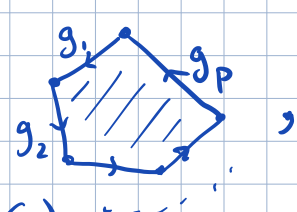


fix G a finite group

a "state" on Σ = decoration of <u>edges</u> by elements of G

$g \nearrow \;\sim\; \swarrow g^{-1}$

s.t. for each face

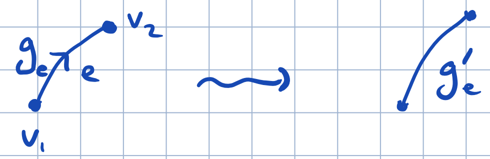


one has $g_p \cdots g_2 g_1 = 1$ (*)

(flatness condition)

Two states $\{g_e\}$, $\{g'_e\}$ are "equivalent" (gauge) if

$$g'_e = h_{v_2}\, g_e\, h^{-1}_{v_1}$$



for all edges, for some
$\{h_v\}$: vertices $\to G$

I.e. we have $G^V \curvearrowright \{G^E\}_{flat}$
gauge transformations   states

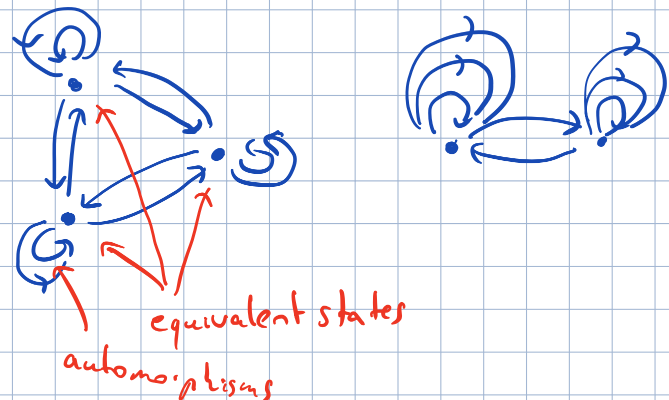
- For $G \subset S_N$, state$/\sim$ = data of an N-leaf covering of Σ with deck transformations in G (a "G-covering"), up to iso

- $\{\text{states}\}/\sim \; = \mathrm{Hom}_{group}(\pi_1(\Sigma, *), G)/G$

$\{g\} \longmapsto (\alpha \mapsto g_{e_s} \cdots g_{e_1})$ – well-defined due to flatness condition (*)
$\uparrow$ closed path along edges $e_1 \cdots e_s$ — group hom. by construction.

- $\{\text{states}\}/\sim$ is a groupoid

$G^V \curvearrowright \{G^E\}_{flat}$
arrows   objects



- Dijkgraaf-Witten partition function:

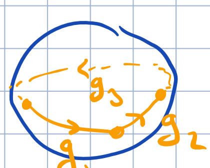
$$Z_\Sigma := \mathrm{Vol}_{gpd}\left(\{\text{states}\}/\sim\right) = \sum_{\substack{\text{non-equiv}\\ \text{states } \{g\}}} \frac{1}{|\mathrm{Aut}|} = \frac{\#\,\text{states}}{|G|^V}$$

arrows (symmetries)

$$= \frac{\#\,\mathrm{Hom}(\pi_1(\Sigma), G)}{|G|}$$

– independent of the cell decomposition of Σ !

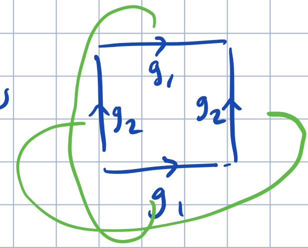
Ex: sphere



$$Z(S^2) = \frac{\#\{(g_1,g_2,g_3) \mid g_3 g_2 g_1 = 1\}}{\#\{(h_1,h_2,h_3)\}} = \frac{|G|^2}{|G|^3} = \frac{1}{|G|}$$

flatness — i.e. $g_3 = g_1^{-1} g_2^{-1}$

Ex: torus



$$Z = \frac{\#\{(g_1,g_2)\in G \mid \overbrace{g_2^{-1}g_1^{-1}g_2 g_1 = 1}^{\text{flatness}}\}}{\#\{h\in G\}}$$

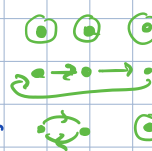
$$= \frac{1}{|G|}\sum_{g_1} \#\underset{\uparrow}{C_{g_1}} = \frac{1}{|G|}\sum_{g_1}\frac{|G|}{\#\underset{\uparrow}{[g_1]}} = \sum_{g_1\in G}\frac{1}{\#[g_1]} = \#\underbrace{G/G}_{\text{set of conjugacy classes}}$$

centralizer of g_1 = all commuting elements

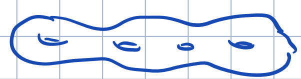
conjugacy class of g_1

Ex: $G = S_3$,

then $Z(S^2) = \frac{1}{6}$, $Z(S^1\times S^1) = \underbrace{3}_{\#\text{ conj. classes}}$



- General formula for the answer



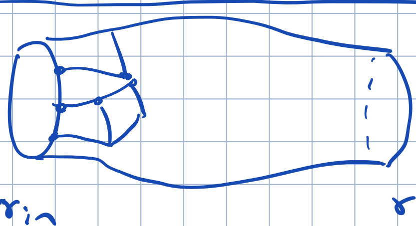
$$Z(\;\cdot\;) = \frac{1}{|G|^{2-2h}}\sum_{R}(\dim R)^{\overbrace{2-2h}^{\text{Euler char. of }\Sigma}}$$

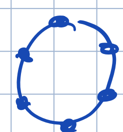
genus h

R: irred. rep. of G

Ex: $G = S_3 \rightsquigarrow Z(\Sigma_h) = \frac{1}{6^{2-2h}}\big(\underset{\text{triv rep}}{1^{2-2h}} + \underset{\boxplus}{2^{2-2h}} + \underset{\text{alt rep}}{1^{2-2h}}\big)$

- Surfaces with boundary

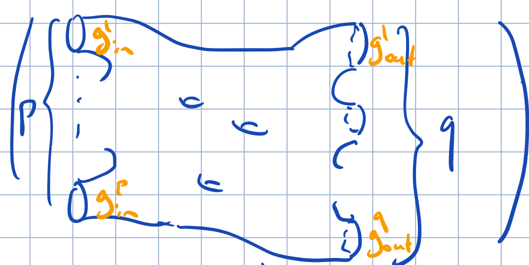




$$\mathcal{H}(\;\cdot\;) = \mathrm{Span}_{\mathbb{C}}\{\text{states on } S^1\}/\sim$$

$$= \mathbb{C}^{G/G} = (\mathbb{C}^G)^{G\text{-invar.}}$$

$\{\psi(g) \text{ s.t. } \psi(hgh^{-1}) = \psi(g)\ \forall h\in G\}$



$$Z(\;\cdot\;) : \psi_{in}(g^1_{in},\dots,g^p_{in}) \longmapsto$$

$$\longmapsto \psi_{out}(g^1_{out},\dots,g^q_{out}) = \sum_{g^1_{in}\cdots g^p_{in}} \overbrace{\boxed{\frac{\#\text{ states on }\Sigma\text{ restricting to } g_{in}, g_{out}}{(\#\text{ bulk symmetries})\cdot(\#\text{ in-bdry symmetries})}}}^{Z_{\Sigma_h}(g_{out},g_{in})} \times \psi_{in}(g^1_{in},\dots,g^p_{in})$$

$$Z_{\Sigma_h}(g_{out},g_{in}) = \frac{1}{|G|^{2-2h-p}}\sum_{R}(\dim R)^{2-2h-p-q}\,\overline{\chi_R(g^1_{in})}\cdots\overline{\chi_R(g^p_{in})}\cdot\chi_R(g^1_{out})\cdots\chi_R(g^q_{out})$$

character

(Gluing property follows from Schur's orthogonality)

- Which Frobenius algebra corresponds to this 2d TQFT?
  - It is $\boxed{Z(\underbrace{\mathbb{C}[G]}_{\text{group ring (convolution ring)}})}$ center

$(\alpha * \beta)(g) = \sum \alpha(h)\beta(h^{-1}g)$

* **2d Yang-Mills theory**

G = compact Lie group, e.g. $G = SU(2)$

Σ - surface

classical theory: field $A \in \mathrm{Conn}_{\Sigma,G} \simeq \Omega^1(\Sigma, \mathfrak{g})$ — for G simply-connected; $\mathfrak{g}$ = Lie(G)

action functional: $S[A] = \frac{1}{2}\int_\Sigma \mathrm{tr}\, F_A \wedge * F_A$

curvature, $F_A = dA + \frac{1}{2}[A,A]$

Quantum picture:

space of states $\mathcal{H}_{S^1} = L^2(G)^G$ - class functions on G

- there is an orthonormal basis of characters $\{\chi_R(U)\}_{R \in \mathrm{Irrep}(G)}$

partition function (Migdal-Witten)

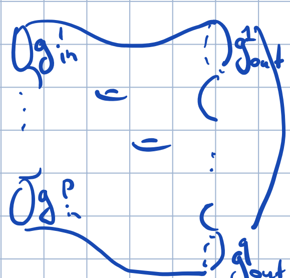
for a closed surface: $Z($  $)$ = " $\int_{\mathrm{Conn}_\Sigma} dA\, e^{-S[A]}$ " ✓ functional integral

genus h

$$= \sum_R (\dim R)^{2-2h} \cdot e^{-a \cdot C_2(R)}$$

a: area(Σ); $C_2(R)$: quadratic Casimir

with boundary:

$Z\Big($  $\Big)$

$$= \text{"} \int_{\mathrm{Conn}_\Sigma} dA\, e^{-S[A]} \text{"} = \sum_R (\dim R)^{2-2h-p-q} \cdot e^{-a\, C_2(R)} \cdot \prod_{i=1}^{p} \overline{\chi(g_{in}^i)} \cdot \prod_{j=1}^{q} \chi(g_{out}^j)$$

Conn_Σ with holonomies around bdry circles being $g_{in}^1 \ldots g_{in}^p$, $g_{out}^1 \ldots g_{out}^q$