Feb 17 talk at ND GSS

## On the Fukaya-Morse $A_\infty$ category

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Plan:
- Fukaya-Morse $A_\infty$ category
- Picture (1a): homotopy transfer
- (1b): effective field theory
- (2): TQM

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### Morse complex (reminder)

$(X, g)$ - cpt. Riemannian mfd ,

$F$ - a Morse function on $X$ , $v = -\mathrm{grad}(F)$ gradient vector field

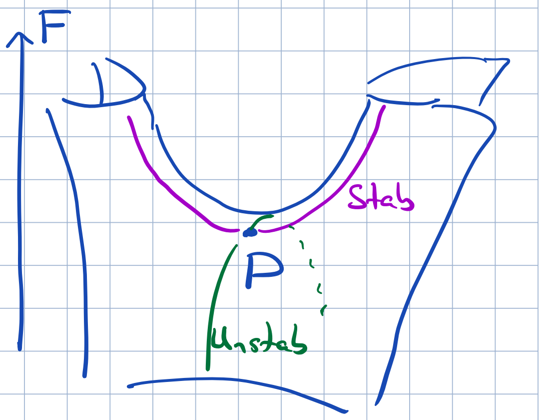
for $P \in \mathrm{crit}(F)$ ,

$$\underset{\uparrow}{\mathrm{Stab}_P} = \{x \in X \mid \mathrm{Flow}_t(v)\circ x \xrightarrow[t\to+\infty]{} P\}$$

open disk of dim = coind(P) = n-ind(P)

$$\underset{\uparrow}{\mathrm{Unstab}_P} = \{x \in X \mid \mathrm{Flow}_t(v)\circ x \xrightarrow[t\to-\infty]{} P\}$$

open disk of dim = ind(P)



$$\mathcal{M}(P,Q) = \frac{\mathrm{Unstab}_P \cap \mathrm{Stab}_Q}{x \sim \mathrm{Flow}_t(v)\circ x} \text{ , for any } t \in \mathbb{R}$$

↑ moduli space of grad trajectories

$$\dim \mathcal{M}(P,Q) = \mathrm{ind}\,P - \mathrm{ind}\,Q - 1$$

Morse chain complex:

$$MC_k(X,F) = \mathrm{Span}_{\mathbb{Z}}(\mathrm{crit}_{\mathrm{ind}=k}(F))$$

differential $d_{\mathrm{Morse}} : MC_k \longrightarrow MC_{k-1}$

$$P \longmapsto \sum_{Q\in \mathrm{crit}_{k-1}(F)} \underset{\uparrow}{\#}\mathcal{M}(P,Q)\cdot Q$$

oriented count of points

- $MC_\bullet(X,F) \simeq C_\bullet(\underset{\nearrow}{X_F})$

CW complex, $\{\text{cells}\} = \{\mathrm{Unstab}_P\}_{P\in \mathrm{crit}(F)}$

## Fukaya-Morse $A_\infty$ category (Fukaya '93, Fukaya-Oh '97)

$X$ - cpt. Riem. mfd

**Ob:** functions $F_1, \dots, F_N$ on $X$

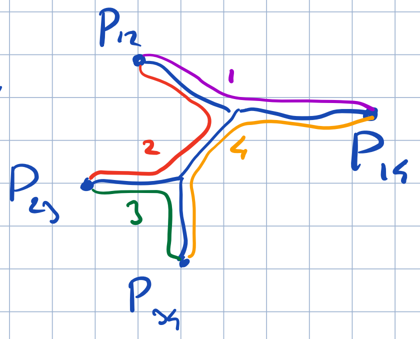
cohomological grading by $\mathrm{coind}(P) = n - \mathrm{ind}(P)$

**Mor:** $\mathrm{Mor}(F_a, F_b) = \mathrm{span}_{\mathbb{Z}}(\mathrm{crit}(F_a - F_b)) = MC^\bullet(F_a - F_b)$

(higher)
**compositions:** $\mu : \mathrm{Mor}(F_1, F_2) \otimes \mathrm{Mor}(F_2, F_3) \otimes \dots \otimes \mathrm{Mor}(F_{N-1}, F_N) \to \mathrm{Mor}(F_1, F_N)$

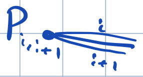
$$\mu(P_{12}, P_{23}, \dots, P_{N-1\,N}) = \sum_{P_{1N} \in \mathrm{crit}(F_1 - F_N)} \# \mathcal{M} \cdot P_{1N}$$

$\mathcal{M}$ = moduli space of "Morse trees" in $X$

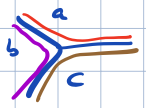


(ab)-edge is a grad. trajectory of $F_a - F_b$

1-val vertices



3-val vertices

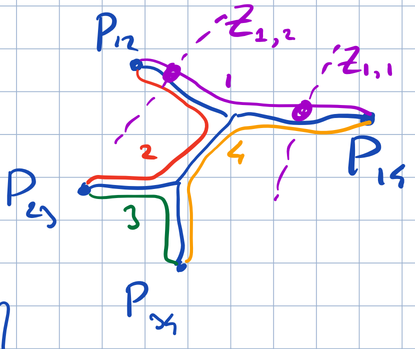


"**Enhancement**": $\mathrm{Mor}(F, F) := C_\bullet^{\mathrm{sing}}(X)$

composition maps: $\mathrm{Mor}(F_1, F_1)^{\otimes k_1} \otimes \mathrm{Mor}(F_1, F_2) \otimes \cdots \mathrm{Mor}(F_{N-1,N}) \otimes \mathrm{Mor}(F_N, F_N)^{\otimes k_N} \to \mathrm{Mor}(F_1, F_N)$

$$\mu(\{Z_{1,\alpha}\}, P_{12}, \dots, P_{N-1\,N}, \{Z_{N,\alpha}\}) = \sum_{P_{1N}} \# \mathcal{M} \cdot P_{1N}$$

$\mathcal{M}$: Morse trees passing through chains $Z_{a,\alpha}$



$$\boxed{\dim \mathcal{M} = -\sum_{a=1}^{N-1} (\mathrm{coind}\, P_{a,a+1} - 1) + (\mathrm{coind}\, P_{1N} - 1) - \sum \mathrm{codim}\, Z_{\alpha,k} - 1}$$

- $\mu_1(P_{12}) = d_{\mathrm{Morse}}(P_{12})$
- $\mu_1(Z) = \partial Z$

$\Big\}$ differentials

- $\mu_2(Z_1, Z_2) = Z_1 \cap Z_2$
- $\underset{n \geq 3}{\mu_n}(Z_1, \dots, Z_n) = 0$

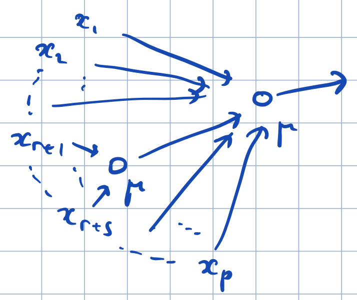
# $A_\infty$ relations

$x_{a_i} \in \mathrm{Mor}(F_{a_i}, F_{a_{i+1}})$ , $i = 1 \dots p$

might coincide

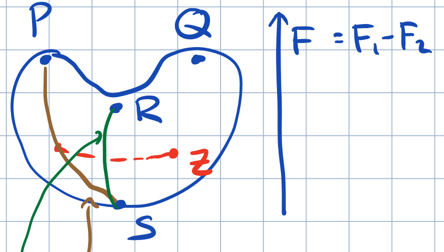
$$\sum_{r,s} \pm\mu(x_1, \dots, x_r, \mu(x_{r+1}, \dots, x_{r+s}), x_{r+s+1}, \dots, x_p) = 0$$

or: $\sum$



$= 0$

**Ex:** $X = S^2$



$x_1 = P \in \mathrm{Mor}(F_1, F_2)$ , $x_2 = Z \in \mathrm{Mor}(F_2, F_2)$

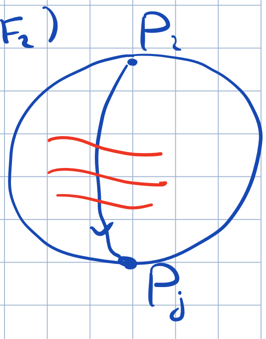
$A_\infty$-rel:

$$\underbrace{\mu(\underbrace{\mu(P)}_{R}, Z)}_{S} + \underbrace{\mu(P, \underbrace{\mu(Z)}_{\partial Z}))}_{-S} + \mu(\underbrace{\mu(P, Z)}_{0 \quad (\dim \mathcal{M} = 1)})) = 0$$

**Ex:** consider compositions $\mu: \mathrm{Mor}(F_1,F_2) \otimes \mathrm{Mor}(F_2,F_2)^{\otimes k} \to \mathrm{Mor}(F_1, F_2)$

$\{P_i\} = \mathrm{crit}(\underset{F_1 - F_2}{F})$

fix $\{C_\alpha\}$ - cycles on $X$



$$m_i^j(T) = \sum_{k \geq 1} \sum_{\alpha_1, \dots, \alpha_k} \#\mathcal{M}(P_i, C_{\alpha_1}, \dots, C_{\alpha_k}, P_j) \cdot \underbrace{T_{\alpha_1} \cdots T_{\alpha_k}}_{\text{generating parameters}},$$

# grad traj $P_i \to P_j$ passing through given cycles

$|T_\alpha| = 1 - \mathrm{codim}\, C_\alpha$

$A_\infty$ relations $\Rightarrow$ $(d_{Morse} + m(T))^2 = 0$

HPT:

$$\begin{tikzcd}
\Omega^\bullet(X) \arrow[loop left, "K"] \arrow[d, shift left, "p"] \\
MC^\bullet(X,F) \arrow[u, shift left, "i"]
\end{tikzcd}$$

, $\underbrace{d}_{d_1} + \underbrace{\sum_\alpha T_\alpha \delta_{C_\alpha}\hat{}}_{d_2 \text{ - deformation of the differential}}$ ← "Novikov differential"

, $d_{Morse} + \underbrace{m(T)}_{\text{induced deformation of the differential}}$

$p d_2 i - p d_2 K d_2 i + \dots$

"Morse contraction"

$$\begin{tikzcd} \Omega^\bullet(X) \arrow[loop left, "K"] \arrow[d, shift left, "p"] \\ MC^\bullet(X,F) \arrow[u, shift left, "i"] \end{tikzcd}$$

$i: \underset{\text{crit pt}}{P} \longmapsto \delta_{\text{Unstab}(P)}$

$p: \omega \longmapsto \sum_P \left( \int_X \omega \wedge \delta_{\text{Stab}(P)} \right) \cdot P$

$$K = \int_0^\infty dt\, \iota_{\underset{\substack{\uparrow \\ \text{grad.v.f. of } F}}{v}} e^{-t\mathcal{L}_v} \quad : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$$

int. kernel: $\delta_Y \in \Omega_{\text{distr}}(X \times X)$ , $Y = \{(x,y) \mid x = \text{Flow}_t(v) \circ y \text{ for some } t>0 \}$

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## Picture (1a)

## Homotopy transfer

$$\begin{tikzcd} V \arrow[loop left, "\underline{K}"] \arrow[d, shift left, "\underline{p}"] \\ V' \arrow[u, shift left, "\underline{i}"] \end{tikzcd}$$

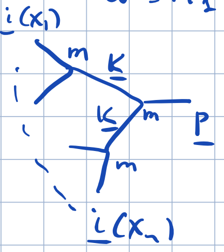
$V, d, m$ ← dga

$V'$ - def. retract

$d' = m'_1, m'_2, m'_3, \dots$ - induced $A_\infty$ algebra structure

$$m'_n(x_1, \dots, x_n) = \sum_{\substack{\text{binary} \\ \text{rooted trees } T \\ \text{with } n \text{ leaves}}}$$

$x_i \in V'$



Kontsevich-Soibelman sum-over-trees formula

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**Fukaya-Morse case:**

$$\underline{K} \circlearrowright V = \Omega^\bullet(X) \otimes \text{Mat}_{\underset{\substack{\downarrow \\ \text{if } Ob = \{F_1, \dots, F_N\}}}{N \times N}} = \bigoplus_{a,b=1}^{N} \Omega^\bullet_{ab}(X) \quad \text{-dga}$$

$$\begin{tikzcd} V \arrow[d, shift left, "\underline{p}"] \\ \mathbb{M} \arrow[u, shift left, "\underline{i}"] \end{tikzcd}$$

$\mathbb{M} = \bigoplus_{a,b} \mathbb{M}_{ab}$

$$\mathbb{M}_{ab} = \begin{cases} MC(F_a - F_b), & a \neq b \\ \Omega^\bullet_{aa}, & a = b \end{cases}$$

$$(\underline{i}, \underline{p}, \underline{K}) = \begin{cases} \text{Morse contraction for } F_a - F_b, & a \neq b \\ \text{trivial } (\underline{i} = \underline{p} = \text{id}, \underline{K} = 0), & a = b \end{cases}$$

Then: induced $A_\infty$ algebra on $\mathbb{M} \rightleftarrows$ F-M $A_\infty$ category

$$m(x_1 \otimes e_{a_1 a_2}, x_2 \otimes e_{a_2 a_3}, \dots, x_p \otimes e_{a_p a_{p+1}}) = \mu(x_1, \dots, x_p) \otimes e_{a_1 a_{p+1}}$$

$e_{ab} \in \text{Mat}_{N \times N}$ - matrix with $(ab)$-entry $1$
all other entries $0$

**Picture (1b)** "BF-theory" $\mathcal{F} = \Omega^\bullet(X)\otimes \mathrm{Mat}_{N\times N}[1] \oplus \Omega^\bullet(X)\otimes \mathrm{Mat}_{N\times N}[n-2]$

$\ni (A, B)$

$$S = \int_X \langle B \overset{\wedge}{,} dA + \tfrac{1}{2}[A,A] \rangle$$

$$A = \begin{pmatrix} A_{11} & A_{12} & \vdots \\ A_{21} & A_{22} & \vdots \\ \cdots & \cdots & \ddots \end{pmatrix}$$ - matrix-valued dif. form

integrate out the complement

$$A' = \begin{pmatrix} \ddots & & \\ & \ddots & 0 \\ & & \ddots \end{pmatrix}$$

d.f. forms

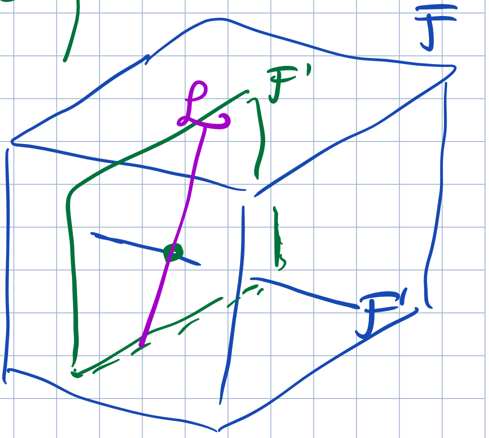
Morse dein of Fa-Fb

Splitting

$$\mathcal{F} \simeq \underbrace{T^*[-1](\mathbb{M}[1])}_{\mathcal{F}'} \oplus \mathcal{F}''$$

$\mathcal{F}'$ "slow fields" $\ni (A', B')$

$\uparrow$ "fast fields"



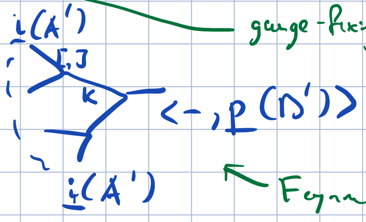
$$e^{\frac{i}{\hbar}S^{\mathrm{eff}}(A',B')} = \int_{\mathcal{L}\subset\mathcal{F}''} e^{\frac{i}{\hbar}S(A'+A'', B'+B'')}$$

"$\mathrm{im}\,\underline{K}$" Lagr

"fiber BV integral"

gauge-fixing $\boxed{\iota_{v_{ab}} A_{ab} = 0}$

$$S^{\mathrm{eff}}(A', B') = \sum_{\text{binary rooted trees}}$$



$\leftarrow$ Feynman diagrams

$$= \sum_{k\geq 1} \frac{1}{k!}\langle B', l_k(A', \ldots, A')\rangle$$

$l_k : \Lambda^k \mathbb{M} \longrightarrow \mathbb{M}$ $\quad$ **$L_\infty$-brackets**

(actually, need to replace $\mathbb{M} \to \mathbb{M}\otimes\mathcal{A}$, $\Omega^\bullet_X \otimes \mathrm{Mat} \otimes \underline{\mathcal{A}}$)

$\uparrow$ upper-triang $\tilde{N}\times\tilde{N}$ matrices

$L_\infty$ relations $\Leftrightarrow$ $\{S^{\mathrm{eff}}, S^{\mathrm{eff}}\} = 0$

BV master equation

# Picture 2: TQM

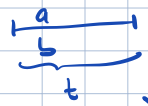
space of states for a particle of $(ab)$-type, $a \neq b$ $\quad \mathcal{H}_{ab} = \Omega^\bullet(X)$, $\quad Q = d$ "BRST operator"

$G = \iota_{v_{ab}}$

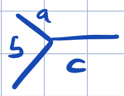
Hamiltonian: $H = \mathcal{L}_{v_{ab}} = [Q, G]$

Evolution operator (propagator) $\quad U(t, dt) = e^{-tH - dt\, G} \in \underset{\substack{\uparrow \\ \text{mod space of metric intervals}}}{\Omega^\bullet(\mathbb{R}_+)} \otimes \mathrm{End}(\mathcal{H}_{ab})$

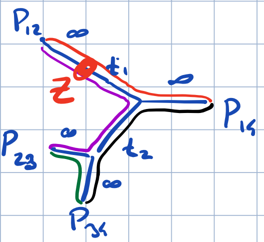
## TQM on metric trees:

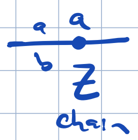
edge


$\sim \quad U(t, dt)$

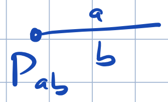
3-val vertrex


$\sim \quad \mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \xrightarrow{\wedge} \mathcal{H}_{ac}$



2-val vertex


$\sim$ operator $\quad \mathcal{H}_{ab} \xrightarrow{\wedge \delta_Z} \mathcal{H}_{ab}$

1-val vertex


$\sim$ state $\quad \delta_{U_{unstab}(P_{ab})}$

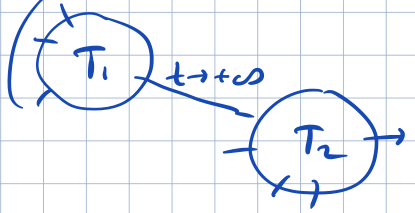
$MT_{N; k_1, \ldots, k_N}$ = moduli space of metric trees with $N$ colors, $k_a = \#\left\{ \begin{matrix} \text{2-val. vertices} \\ \text{of color } a \end{matrix} \right\}$

$$\underset{\substack{\uparrow \\ \text{product of decorations}}}{I} \in \Omega^\bullet(MT_{N; k_1, \ldots, k_N}) \otimes \mathrm{Hom}(Mor_{1,1}^{\otimes k_1} \otimes Mor_{1,2} \cdots Mor_{N-1,N} \otimes Mor_{N,N}^{\otimes k_N}, Mor_{1,N})$$

$Mor_{a,b} = Mor(F_a, F_b)$

**Properties:** (1) $(d_{MT} + Q) I = 0$

(2) Factorization on IR-bdry of MT:

$$I\Big($$


$$\Big) = \langle I(T_2) \overset{\wedge}{;} I(T_1) \rangle$$

(3) $\displaystyle\int_{MT} I = \mu$ - composition map in F-M category

# TQM proof of $A_\infty$ relations

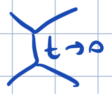
$$(1) \Rightarrow \int_{MT} (d_{MT} + Q)\, I = 0$$

$$\Rightarrow \underbrace{\int_{\partial MT} I}_{/\!/} = \boxed{Q \int_{\underbrace{MT}_{\mu}} I}$$

terms $\mu(\cdot \cdot \cdot \mu(\cdot) \cdots\cdots)$ in $A_\infty$ relations

$$\underbrace{\sum_{\substack{\text{bdy strata}\\ \text{of } MT\\ \text{with } t_{edge} \to 0}} \int_{\text{stratum}} I}_{0} + \underbrace{\sum_{\substack{\text{bdy strata}\\ \text{with } t_{edge} \to +\infty}} \int_{\text{stratum}} I}_{\text{terms } \mu(\cdots \mu(\cdots) \cdots) \text{ in } A_\infty \text{ relations}}$$

when summed over trees. (cancellations,

like  +  $= 0$ )