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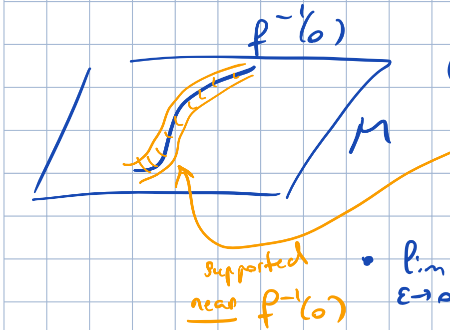
Mathai-Quillen formalism and A model in D=1,2

Mathai-Quillen formalism

- special case : $f: M \to \mathbb{R}$ manifold — assume 0 is a regular value of f

  how to write a (smeared) δ-form supported on $f^{-1}(0)$?



$$(1)\quad \boxed{\delta^{\varepsilon}_{f^{-1}(0)} = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{1}{2\varepsilon} f(x)^2} df} \qquad \in \Omega^1_{cl}(M)$$

- $\lim_{\varepsilon \to 0} \int_M \delta^{\varepsilon}_{f^{-1}(0)} \wedge \underset{\in \Omega^{n-1}(M)}{\alpha} = \int_{f^{-1}(0)} \alpha$
- if $\alpha \in \Omega^{n-1}_{cl}(M)$, then works for any ε, without having to take the limit.

- since the class $[\delta^{\varepsilon}_{f^{-1}(0)}] \in H^1_{\text{de Rham}}(M)$ is independent on ε

Rewriting (1):

$$\delta^{\varepsilon}_{f^{-1}(0)} = \int_{\mathbb{R}[-1]} D\pi \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{1}{2\varepsilon} f(x)^2 + \pi\, df}$$

π: odd variable (degree = -1)

$$= \frac{i}{2\pi} \int_{\mathbb{R}[-1] \oplus \mathbb{R}} D\pi\, dp\, e^{ipf(x) - \frac{\varepsilon p^2}{2} - i\pi\, df} = \frac{i}{2\pi} \int_{T[1]\underbrace{\mathbb{R}[-1]}_{V_{aux}}} D\pi\, dp\, e^{(d + \overbrace{p\frac{\partial}{\partial \pi}}^{d_{aux}})(i\pi f(x) - \frac{\varepsilon \pi p}{2})}$$

p: even variable (deg = 0)

General setup

(oriented) vector bundle

$$\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$$

E (usually one assumes $\mathrm{rk}\, E = 2m$ even), g fiber metric, ∇ connection compatible with g, $s: M \to E$ section

MQ (representative for the Euler class)

$g^{bb'} \pi_{b'} F^a{}_b \pi_a$

$\pi_a F^a{}_b g^{bb'} \pi_{b'}$

$$\delta^{\varepsilon}_{s^{-1}(0)} = \frac{1}{(i\sqrt{2\pi\varepsilon})^{\mathrm{rk}E}} \int \underset{\text{fiber Berezinian induced from } g}{D\pi}\, e^{\overbrace{-\frac{1}{2\varepsilon} g(s,s) - i\langle \pi, \nabla s\rangle - \frac{\varepsilon}{2}\langle \pi, F_\nabla(g^{-1}(\pi))\rangle}^{S_{MQ} \in \Omega(M, \wedge E)}} \qquad (2)$$

$$\Xi_{E,\nabla,s} \qquad \text{fiber integral over fibers of } \begin{tikzcd} E^*[-1] \arrow[d] \\ M \end{tikzcd}$$

integrand

$$\in \Gamma(M, \underbrace{\Lambda^\bullet E}_{\text{fun. of } \pi} \otimes \underbrace{\Lambda^\bullet T^* M}_{\text{form on } M} \otimes \underbrace{\Lambda^{\mathrm{rk} E} E^*}_{\text{fiber Berezinian } D\pi})$$

Berezin integral

- $\Xi_{E,\nabla,s,\frac{g}{\varepsilon}} \in \Omega^{\mathrm{rk} E}_{\underline{\text{closed}}}(M)$

-its class in $H^{\mathrm{rk} E}(M)$ is independent of $s, \nabla, g, \varepsilon$

and is the Euler class of $\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$

- $\varepsilon \to 0$: $\Xi \to \delta$-form on $s^{-1}(0)$

- $\varepsilon \to \infty$ (or $s=0$) : $\Xi \to \frac{1}{(2\pi)^{\frac{1}{2}\mathrm{rk} E}} \mathrm{Pf}(F_\nabla)$

- **Ex:** $E = TM$; s generic -vector field $, \nabla = \nabla_{\text{Levi-Civita}}$

(i.e. intersecting zero-section $M \subset TM$ transversally)

$$\lim_{\varepsilon\to 0} \int_M \Xi = \lim_{\varepsilon\to\infty} \int_M \Xi$$

$$\sum_{\substack{\text{zeroes } x_i \\ \text{of } s}} \underbrace{\pm 1}_{\mathrm{ind}_{x_i}(s)} \qquad\qquad \frac{1}{(2\pi)^{\mathrm{rk} E/2}} \int \mathrm{Pf}(\underbrace{R}_{F_{\nabla_{LC}}})$$

Poincaré -Hopf thm $\;\; \chi(M) \;\;$ Chern-Gauss-Bonnet thm

- Rewriting (2):

canonical fiber integration measure $\quad p = \text{“}d\pi\text{”}$

$$\Xi = \frac{1}{(2\pi i)^{\mathrm{rk} E}} \int_{E^* \oplus E^*[-1] \to M} \boxed{D\pi \, d\boxed{p}} \, e^{\overbrace{(d_M + \langle p, \frac{\partial}{\partial \pi}\rangle)\left(i\langle \pi, s(x)\rangle - \frac{\varepsilon}{2}\langle \pi, g^{-1}(p) - A(g^{-1}(\pi))\rangle\right)}^{i\langle p, s(x)\rangle \overset{-i\langle \pi, ds\rangle}{-} \frac{\varepsilon}{2} g^{-1}(p,p) + \varepsilon\langle p, A(g^{-1}(\pi))\rangle - \frac{\varepsilon}{2}\langle \pi, dA(g^{-1}(\pi))\rangle}} \qquad (3)$$

$$= \frac{1}{(2\pi i)^{\mathrm{rk} E}} \int e^{d_{E^*[-1]}\left(i\langle \pi, s(x)\rangle - \frac{\varepsilon}{2} g^{-1}(\pi, \tilde{A})\right)}$$

pushforward of a form along $\begin{tikzcd} E^*[-1] \arrow[d] \\ M \end{tikzcd}$

or: pushforward of a function along $\begin{tikzcd} T[1](E^*[-1]) \arrow[d] \\ T[1]M \end{tikzcd}$

$$\overset{\text{“}d\pi + A^T(x)\pi\text{”}}{\tilde{A}} \in \Omega^1(E^*[-1], T^{\mathrm{vert}} E^*[-1])$$

← in a local trivialization

↑ connection 1-form for the connection dual to ∇

Limit $\varepsilon \to 0$:

$$\underset{\substack{\shortparallel \\ \delta_{s=0}}}{\Xi_s^{\varepsilon=0}} = \left(\frac{i}{2\pi}\right)^{\mathrm{rk}E} \int_{\substack{\text{fibers of } E^*\oplus E^*[-1] \\ \downarrow \\ M}} D\pi\, dp \; e^{i\langle p, s(x)\rangle - i\langle \pi, ds\rangle}$$

Then: for $\theta \in \underline{\Omega^{n-\mathrm{rk}E}_{\text{closed}}}(M)$,

$$\underbrace{\int_M \Xi^{\varepsilon}_{E,\nabla,s} \wedge \theta}_{\substack{\text{indep. of } \varepsilon\text{, can take} \\ \text{the limit } \varepsilon \to 0}} = \int_{s^{-1}(0)} \theta$$

<u>Rem</u> An alternative way to write Ξ as $\int D\pi\, dp$:

$$\Xi = \left(\frac{i}{2\pi}\right)^{\mathrm{rk}E} \int_{\substack{E^*\oplus E^*[-1] \\ \downarrow \\ M}} D\pi\, dp\; e^{i\langle p, s\rangle - i\langle \pi, \nabla s\rangle - \frac{\varepsilon}{2} g^{-1}(p,p) - \frac{\varepsilon}{2}\langle \pi, F(g^{-1}(\pi))\rangle} \tag{4}$$

$$= \left(\frac{i}{2\pi}\right)^{\mathrm{rk}E} \int D\pi\, dp\; e^{\underbrace{\left(d_M + \langle \overset{p+A^T\pi}{\overbrace{\tilde{A}}}, \frac{\partial}{\partial \pi}\rangle + \overset{\text{something like that...}}{\overbrace{\langle F(\pi) + Ap, \frac{\partial}{\partial p}\rangle}}\right)}_{Q \,\leftarrow\, \text{a } \underline{\text{cohom.}} \text{ vector field}} \left(i\langle \pi, s(x)\rangle - \frac{\varepsilon}{2} g^{-1}(\pi, p)\right)}$$

(cf. Benechi-Cattaneo-Iraso (13), (14),
Siye Wu (4.2))

Aside

Representative of the Thom class

integral form on $E\oplus E^*[-1]$

$$\mathrm{Th} = \frac{1}{(-i\sqrt{2\pi\varepsilon})^{\mathrm{rk}E}} \int_{\text{fiber of } \begin{tikzcd} E\oplus_M E^*[-1] \arrow[d] \\ E \end{tikzcd}} D\pi \; e^{-\frac{1}{2\varepsilon} g(\xi,\xi) - i\langle \pi, \underbrace{\mathcal{A}}_{\substack{d\xi + A(x)\xi \\ \in \Omega^1(E,\, T^{vert}E) \\ \text{-1-form of } \nabla}} \rangle - \frac{\varepsilon}{2}\langle \pi, F(g^{-1}(\pi))\rangle} \in \Omega^{\mathrm{rk}E}_{cl}(E)$$

notations: $(x, \xi, \pi) \in E \oplus_M E^*[-1]$

$x \in M$, $\xi \in E_x$, $\pi \in E^*_x[-1]$

> Th is a closed form representing the Thom class of $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$

- for $s : M \to E$ a section, $\underbrace{\Xi_{E,s,\nabla,g}}_{\substack{\text{rep. of} \\ \text{Euler class}}} = s^* \underbrace{\mathrm{Th}_E}_{\substack{\text{rep. of} \\ \text{Thom class}}}{}_{,\nabla,g}$

- $$\mathrm{Th} = \frac{1}{(2\pi i)^{\mathrm{rk}E}} \int_{\text{fiber of } \begin{tikzcd} T[1](E\oplus E^*[-1]) \arrow[d] \\ T[1]E \end{tikzcd}} e^{d_{E\oplus E^*[-1]}\left(i\langle \pi, \xi \rangle - \frac{\varepsilon}{2} g^{-1}(\pi, \tilde{\mathcal{A}}) \right)}$$

"1D A-model"

Quantum picture: M cpt. mfd, v - vector field on M

HTQM with $\underset{\text{space of states}}{\mathcal{H}=\Omega^\bullet(M)}$, $Q=d$, $G=\iota_v$ $\rightsquigarrow$ $H=[d,\iota_v]=\mathcal{L}_v$

– Hamiltonian

observables $\underset{\underset{\omega\wedge}{\parallel}}{O_\omega}$ for $\omega\in\Omega^\bullet(M)$

$\boxed{\text{evol. operator: } I_{t,dt}=(1+dt\,\iota_v)\cdot\mathrm{Flow}_t(v)}$

Notation:
$\gamma=[t_{in},t_{out}]$

* I think:
we need $\omega_1\ldots\omega_n$ to be supported away from c_{in}, c_{out}

$$I=\underbrace{\langle\omega_{out}|}_{\delta_{c_{out}}}O_{\omega_n}\big|_{t_n,dt_n}\cdots O_{\omega_1}\big|_{t_1,dt_1}\underbrace{|\omega_{in}\rangle}_{\delta_{c_{in}}}\in\Omega^\bullet(\mathrm{Conf}_n(\gamma))$$

– closed form if ω's are closed

$$\parallel$$

$$\int_{\substack{v\text{-traj}\\ x(t_{in})=c_{in}\\ x(t_{out})=c_{out}}} ev_1^*\omega_1\wedge\cdots ev_n^*\omega_n$$

← localized path integral formula

$$v\text{-traj}=\left\{\text{maps } x:\gamma\to M \text{ s.t. } \frac{dx}{dt}=v(x)\right\}$$

$$v\text{-traj}\times\mathrm{Conf}_n(\gamma)\xrightarrow{ev_k}M$$

evaluation at k-th point

$$\int_M\omega_{out}\wedge\left(I_{t_{out}\leftarrow t_n,dt_n}\,\omega_n\,I\cdots\omega_2\,\underbrace{I_{t_2,dt_2\leftarrow t_1,dt_1}}_{\text{HTQM evolution operator}}\,\omega_1\,I_{t_1,dt_1\leftarrow t_{in}}\,\omega_{in}\right)$$

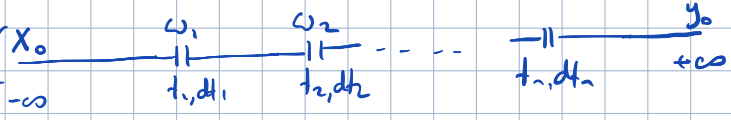
← operator formalism formula

Enumerative meaning of the amplitude:

$$\int_{\mathrm{Conf}_n(\gamma)}I=\#\left\{v\text{-trajectories starting on } \overset{\overset{\text{time}}{\text{at } t_{in}}}{\vee}\!\!c_{in}\text{, ending on } \overset{\text{at time } t_{out}}{\vee}\!\!c_{out} \text{ and passing through } c_1,\ldots,c_n\right\}$$

Special case: $v=\mathrm{grad}(f)$, f - Morse function (M $\overset{\text{cpt,}}{\text{Riemannian}}$)

Can consider $I\Bigg($



$\Bigg)$

$$\in\Omega^\bullet(\mathrm{Conf}_n(\mathbb{R})/\mathbb{R})$$

← translation invariance

x_0, y_0 – crit. points of f ⇝ special asymptotic forms $\omega^{in}_{x_0}$, $\omega^{out}_{y_0}$

- In Morse case, one has a deformation / regularization

$$Q = d, \qquad G^\varepsilon = \iota_{\mathrm{grad}(f)} + \varepsilon d^* \qquad (\Rightarrow H = \mathcal{L}_{\mathrm{grad}(f)} + \varepsilon \Delta)$$

Rem: $T^*(T[1]M) \overset{\text{can. diffeo}}{\cong} T[1](T^*[-1]M)$

$\overset{\text{can. anti-symplectomorphism}}{\cong}$ $T^*(T^*[-1]M)$ $\cong$

$T^*E \leftrightarrow T^*E^*$

$T^*N \cong TN$ for N symplectic

Path integral / AKSZ approach

space of BV fields: $\mathcal{F} = \mathrm{Map}(T[1]\gamma, \boxed{T^*(T[1]M)})$ target

$$\begin{array}{cccc} \pi_i & p_i & \theta^i & x^i \\ -1 & 0 & 1 & 0 \end{array}$$ - loc. coordinates on target

$$Q^{\text{target}} = (\underbrace{d_M}_{\text{as a coh. v.f. on } T[1]M})^{\text{cotangent lift}} = \theta \frac{\partial}{\partial x} + p \frac{\partial}{\partial \pi}$$

symp form ω^{target} = can. symp. form on $T^*(\cdots)$

S^{target} = Hamiltonian for Q^{target} $\qquad \overset{\text{"}}{\delta}(p\,\delta x + \pi\,\delta\theta)$

$= \theta^i p_i$

In loc. coords on M:

$$S_{AKSZ} = \int_\gamma \tilde{p}\, d\tilde{x} + \tilde{\pi}\, d\tilde{\theta} + \tilde{\theta}\tilde{p}$$

$$= \int_\gamma p\, dx + \pi\, d\theta + \pi^+ p + \theta x^+$$

$\overset{0}{} \quad \overset{1}{}$ - de Rham deg

$\tilde{x} = \underset{0}{x} + \underset{-1}{p^+}$ ← ghost number

$\tilde{p} = \underset{0}{p} + \underset{-1}{x^+}$

$\tilde{\theta} = \underset{1}{\theta} + \underset{0}{\pi^+}$

$\tilde{\pi} = \underset{-1}{\pi} + \underset{-2}{\theta^+}$

AKSZ superfields

gauge-fixing: $v \in \mathfrak{X}(M)$ a vector field

$$\Psi = \int_\gamma dt\, \underbrace{\pi\, v(x)}$$

Hamiltonian for $\iota_v \in \mathfrak{X}(T[1]M)$; gets quantized to $G = \iota_v$

$T\,\mathrm{Map}(\gamma, N) \cong$
$\cong \mathrm{Map}(\gamma, TN)$

→ $\mathcal{L}_\Psi = \mathrm{graph}\, d\Psi$:
$\sim \mathrm{Map}(\gamma, T^*T[1]M)$
$\cong T[1]\,\mathrm{Map}(\gamma, T^*[-1]M)$

$\pi^{+i} = dt\, v^i(x)$
$x^+_i = dt\, \partial_i v^j(x)\, \pi_j$

$p^+, \theta^+ = 0$

$$S = S_{AKSZ}\Big|_{\mathcal{L}_\Psi} = \int_\gamma dt \big(p(\dot{x} - v(x)) + \pi_i(\dot{\theta}^i - \partial_j v^i \theta^j)\big)$$

$$= \overbrace{d_{\mathrm{Map}(\gamma, \underset{\pi}{T^*[-1]}\underset{x}{M})}}^{\int \theta \frac{\delta}{\delta x} + p \frac{\delta}{\delta \pi}} \left(\int_\gamma dt\, \pi(\dot{x} - v(x)) \right)$$ as in (3)

$$\int \mathcal{D}p\,\mathcal{D}\pi\; e^{iS} = \text{Mathai-Quillen rep. for } \delta\big(v\text{-traj} \hookrightarrow \mathrm{Map}(\gamma, M)\big)$$

(at $\varepsilon=0$) $\in \Omega^{\bullet}(\mathrm{Map}(\gamma, M))$

$$\begin{tikzcd} \Omega^1(\gamma, x^*TM) \arrow[r] & \mathrm{Map}_0(T[1]\gamma, T[1]M) \arrow[d] & = \underline{E} \\ & \mathrm{Map}(\gamma, M) & = \underline{M} \end{tikzcd}$$

deg=0

$= \Big\{$bundle maps

$$\begin{tikzcd} T\gamma \arrow[r, "\xi"] \arrow[d] & TM \arrow[d] \\ \gamma \arrow[r, "x"] & M \end{tikzcd}$$

$\Big\}$

$\underline{E}$ - bundle of e.o.m.

section: $\underline{M} \longrightarrow \underline{E}$

$x \longmapsto dx - dt\, v(x)$

- observables:

$$\omega \in \Omega^{\bullet}(M) \rightsquigarrow \mathcal{O}^{(0)}_{\omega} = \omega_{i_1\dots i_p}(x)\,\theta^{i_1}\dots\theta^{i_p}$$

$\omega = \omega_{i_1\dots i_p}(x)\,\theta^{i_1}\dots\theta^{i_p}$

$\rightsquigarrow$ "descent tower"

$$\mathcal{O}_{\omega} = \underset{\substack{\uparrow \\ \text{0-form} \\ \text{on } \gamma}}{\mathcal{O}^{(0)}_{\omega}} + \underset{\substack{\uparrow \\ \text{1-form} \\ \text{on } \gamma}}{\mathcal{O}^{(1)}_{\omega}} = \omega_{i_1\dots i_p}(x)\,(\theta^{i_1} + dx^{i_1})\dots(\theta^{i_p} + dx^{i_p})$$

Then:

$$\int_{\mathrm{Map}(\gamma, T^*(T[1]M))} e^{iS}\, \mathcal{O}_{\omega_1}(t_1, dt_1)\dots\mathcal{O}_{\omega_n}(t_n, dt_n) = \int_{v\text{-traj}} ev_1^*(\omega_1)\wedge\dots\wedge ev_n^*(\omega_n) \in \Omega^{\bullet}(\mathrm{Conf}_n(\gamma))$$

$= \text{"}\langle \mathcal{O}_{\omega_1} \cdots \mathcal{O}_{\omega_n} \rangle\text{"}$ - correlator

Remark: In Morse case, $v = -\mathrm{grad}(f)$, we can consider $\varepsilon \neq 0$ case!

Then:

$$\Phi^{\varepsilon} = \int_{\gamma} dt\, \Big(\underbrace{\pi\, v(x) + \frac{\varepsilon}{2}\, g^{-1}(\pi, p + \overbrace{\Gamma\,\theta\,\pi}^{\substack{\text{1-form} \\ \text{connection on } T^*[-1]M}}\ )}_{\text{quantizes to } \iota_v + \frac{\varepsilon}{2} d^*}\Big)$$

$$e^{iS^{\varepsilon}} \xrightarrow[\int \mathcal{D}p]{} e^{\underbrace{-\int_{\gamma} dt \left(\frac{1}{2\varepsilon}(\dot{x} - v(x))^2 + \langle \pi, D_t\theta - (Dv)\theta \rangle + \frac{\varepsilon}{2} R\, \pi\pi\theta\theta\right)}_{S^{\text{2nd order}}}}$$

[Frenkel-Losev-Nekrasov '06 (2.6) "Instantons beyond topol. theory I"]

$x \in \mathrm{Map}(\gamma, M)$

$\theta \in \Gamma(\gamma, x^*TM)$ $\quad gh = +1$ ⎫

$\pi \in \Gamma(\gamma, x^*T^*M)$ $\quad gh = -1$ ⎭ odd

Rem: ~[Witten '82], Morse TQM action has an extra "topological" term: $-\frac{df(x)}{dt}$

$$S^{2nd} \rightsquigarrow S^{2nd} + \frac{1}{\varepsilon}\int dt\, \overbrace{(\dot{x}, v(x))}$$

$f(x_{in}) - f(x_{out})$

2D A model

AKSZ on $\mathrm{Map}(T[1]\Sigma\,,\; T^*[1](T[1]M)\;)$

$$\begin{array}{cccc} \pi & p & \theta & x \\ 0 & 1 & 1 & 0 \end{array}$$

$S_{AKSZ} = \theta p$ as before

$$S_{AKSZ} = \int \tilde{p}\, d\tilde{x} + \tilde{\pi}\, d\tilde{\theta} + \tilde{\theta}\tilde{p}$$

ε-smearing term

choose $\Psi = \int \partial X^i\, \pi_i^{(0,1)} + \bar{\partial} X^{\bar{i}}\, \pi_{\bar{i}}^{(1,0)}$ $\left(+ \frac{\varepsilon}{2} g^{IJ}(x) p_I^{(1)} \pi_J^{(1)} + \ldots \right)$

What to do with fields $p_{\bar{i}}^{1,0}, p_i^{0,1}, \pi_{\bar{i}}^{1,0}, \pi_i^{0,1}$ which do not participate in $(*)$?

- I guess, they can be integrated out after including the "smearing term"

$\mathcal{L}_\Psi$: base: $X\,,\; \pi^{(1)}, p^{(1)}, \theta\,,\; p^{(0)}, \theta^{(2)}$

then $S_{AKSZ}\big|_{\mathcal{L}_\Psi} \xrightarrow[\substack{p^{(0)}, \theta^{(2)},\\ p_i^{0,1}, \pi_i^{0,1} \,\&\, c.c.}]{\text{integrate out}} \int p_i^{(1,0)} \bar{\partial} X^i + p_{\bar{i}}^{(0,1)} \partial X^{\bar{i}} + \pi_i^{(1,0)} \bar{\partial}\theta^i + \pi_{\bar{i}}^{(0,1)} \partial\theta^{\bar{i}}$ $(*)$

$\left(+ \frac{\varepsilon}{2} g^{i\bar{j}} p_i^{(1,0)} p_{\bar{j}}^{(0,1)} + \ldots \right)$

$\to \int D p^{(1)} D\pi^{(1)}\, e^{----}$ = MQ representative for $\{ \mathrm{Hol}(\Sigma, M) \hookrightarrow \mathrm{Map}(\Sigma, M) \}$

bundle maps

$$\begin{tikzcd} T\Sigma \arrow[r, "J"] \arrow[d] & TM \arrow[d] \\ \Sigma \arrow[r] & M \end{tikzcd}$$

fiberwise compatible with J

$$\begin{tikzcd} \Gamma(\Sigma, (T_\Sigma^{0,1})^* \otimes X^* T^{1,0} M) \oplus (T^{1,0}_\Sigma)^* \otimes X^* T^{0,1} M) \arrow[r] & \text{bundle maps} \arrow[d] & = \underline{E} \\ & \mathrm{Map}(\Sigma, M) & = \underline{M} \end{tikzcd}$$

section: $X \in \mathrm{Map}(\Sigma, M) \longmapsto$

$\mapsto (\bar{\partial} X^i, \partial X^{\bar{i}})$

$X^i, X^{\bar{i}}$ - coords on $\underline{M}$

$p_i^{(1,0)}, p_{\bar{i}}^{(0,1)}$ - fib. coords on $\underline{E}^* \to \underline{M}$

$\theta^i, \theta^{\bar{i}}$ - fib coords on $T[1]\underline{M}$

$\pi_i^{(1,0)}, \pi_{\bar{i}}^{(0,1)}$ - fib coords on $\underline{E}^*[-1] \to \underline{M}$

$$(*) = d_{\underline{E}^*[-1]} \Big(\underbrace{\int_\Sigma \pi^{(1)} dX - \Psi}_{\int \pi_i^{1,0} \bar{\partial} X^i + \pi_{\bar{i}}^{0,1} \partial X^{\bar{i}}} \Big)$$

(2nd order)

Standard action of the A-model

(Witten, "lectures on mirror manifolds", 1993)
section 3

Dolbeaux on Σ twisted by the pullback of ∇_{LC} on M

$$S_A = \int_\Sigma \frac{1}{\varepsilon} g_{i\bar j}\, \partial X^{\bar i}\, \bar\partial X^j + \Pi^{(1,0)}_i \bar D \theta^i + \Pi^{(0,1)}_{\bar i} D \theta^{\bar i} + \varepsilon \underbrace{R^{i\bar i}{}_{j\bar j}}_{X^* R_{\nabla_{LC}}} \Pi^{(1,0)}_i \Pi^{(0,1)}_{\bar i} \theta^j \theta^{\bar j}$$

$$+ \int_\Sigma \underbrace{\frac{1}{2\varepsilon} X^* \omega_{\text{Kähler}}}_{\text{top. term}}$$

add up to $\frac{1}{2\varepsilon} g_{IJ} \partial X^I \bar\partial X^J$
- kin. term of nonlin σ-model
(doesn't use target cx structure)

fields

- $X \in \mathrm{Map}(\Sigma, M)$
- $\theta \in \Gamma(\Sigma, X^* TM)$ odd (gh = +1)
- $\Pi^{1,0}_i \in \Omega^{1,0}(\Sigma, X^*(T^{1,0})^* M)$
- $\Pi^{0,1}_{\bar i} \in \Omega^{0,1}(\Sigma, X^*(T^{0,1})^* M)$

$\Big\}$ odd, gh = -1

$$\int D\pi \; e^{-S_A} \in \Omega^\bullet(\mathrm{Map}(\Sigma, M))$$

- MQ-rep for the Euler class/ ε-smeared δ-form on $\mathrm{Hol}(\Sigma, M) \subset \mathrm{Map}(\Sigma, M)$.

Observables: for $\omega \in \Omega^\bullet(M)$

$$\mathcal{O}_\omega = ev^* \omega \in \Omega^\bullet(\mathrm{Map}) \otimes \Omega^\bullet(\Sigma),$$

$$\mathrm{Map}(\Sigma, M) \times \Sigma \longrightarrow M$$

0, 1, 2-form on Σ

$$\mathcal{O}_\omega = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)}$$

$$= \omega_{i_1 \cdots i_p}(X)(\theta^{i_1} + dX^{i_1}) \cdots (\theta^{i_p} + dX^{i_p})$$

$$\langle \mathcal{O}_{\omega_1} \cdots \mathcal{O}_{\omega_n} \rangle = \int_{\mathrm{Map}(\Sigma, M)} DX D\theta \int D\pi \; e^{S_A} \mathcal{O}_{\omega_1} \cdots \mathcal{O}_{\omega_n} \underset{\varepsilon \to 0}{=} \int_{\mathrm{Hol}(\Sigma, M)} ev_1^* \omega_1 \wedge \cdots \wedge ev_n^* \omega_n \in \Omega^\bullet(\mathrm{Conf}_n(\Sigma))$$

representative of Gromov-Witten class (if ω_k are closed forms)

- top. term $\int X^* \omega_{\text{Kähler}}$ in S_A is responsible for weighting holom. maps with q^d.