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mini-course at CMND'24 (grad week, June 3–7, 2024)

Combinatorial models of TQFT

Plan:

- simplicial BF theory
  - A_∞ algebra on $C^0(I)$
  - continuum dual BF
  - abstr. BF and eff. action on a subcomplex
  - simpl. BF

  0809.1160 [+ hep-th/0610326]

  [a.k.a. Effective BV action induced on Whitney forms]

- ~~cellular BF with Segal-like gluing: BV-BFV formalism~~ ~~or: "relative cell BF theory"~~
  - ~~abelian~~
  - ~~non-abelian, by induction in cell dimension~~

  ~~[1701.05874 with A.S. Cattaneo, N. Reshetikhin]~~

- ~~[simplicial 1d Chern-Simons] [1005.2111 with A. Alekseev]~~

- combinatorial 2d TCFT [2402.04468 with J. Beck, A. Losev]

OR

- ~~Fukaya-Morse A_∞ category and BF theory in "CN" gauge [2112.12756 with O. Chekeres, A. Losev, D. Youmans]~~

Simplicial BF theory (effective BV action induced on Whitney forms)

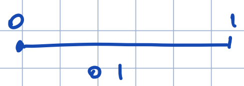
Ref: P.M. "Discrete BF theory" arXiv: 0809.1160

Motivation: de Rham cdga of an interval

"discrete algebra of the interval"

$\Omega^\bullet([0,1]), d, \wedge$

cell cochains



try to model de Rham cdga here

$\Omega^\bullet_{discr} = \mathrm{Span}_{\mathbb{R}}\{e_0, e_1, e_{01}\} = \{x^0 e_0 + x^1 e_1 + x^{01} e_{01}\}$

deg = 0 0 1

with $d: \; e_1 \mapsto e_{01}$, $e_0 \mapsto -e_{01}$

$$\wedge: \quad \begin{array}{c|ccc} & e_0 & e_1 & e_{01} \\ \hline e_0 & e_0 & 0 & \frac{1}{2}e_{01} \\ e_1 & 0 & e_1 & \frac{1}{2}e_{01} \\ e_{01} & \frac{1}{2}e_{01} & \frac{1}{2}e_{01} & 0 \end{array}$$

$\leftarrow$ fixed by:
- supercommutativity
- symmetry of the interval (compatibility w.r.t. reflection)
- $e_0 + e_1 = 1$ is a unit for $\wedge$.

properties: $d^2 = 0$ ✓

$d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$ ✓

$\boxed{(\alpha \wedge \beta) \wedge \gamma \neq \alpha \wedge (\beta \wedge \gamma)}$!

associativity fails !!

E.g. $\overbrace{(e_0 \wedge e_0)}^{e_0} \wedge e_{01} = \frac{1}{2} e_{01}$ $\not\parallel$

$e_0 \wedge \underbrace{(e_0 \wedge e_{01})}_{\frac{1}{2}e_{01}} = \frac{1}{4} e_{01}$

However, associativity can be restored in "homotopical sense":

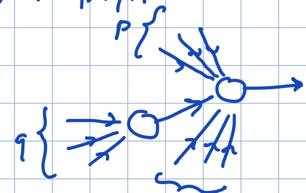
can find $m_3: \Omega^{\bullet\,\otimes 3}_{discr} \to \Omega_{discr}$ s.t.

$$(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 - \alpha_1 \wedge (\alpha_2 \wedge \alpha_3) = -d\, m_3(\alpha_1, \alpha_2, \alpha_3) - m_3(d\alpha_1, \alpha_2, \alpha_3) \pm m_3(\alpha_1, d\alpha_2, \alpha_3) \pm m_3(\alpha_1, \alpha_2, d\alpha_3)$$

In fact, one can extend $d = m_1$, $\wedge = m_2$ to an A_∞ algebra $\{m_k\}_{k \geq 1}$ satisfying

$\uparrow$ k-linear operations

$$\sum_{\substack{p+q+r=n \\ p,r \geq 0,\; q \geq 1}} \pm m_{p+r+1}(\alpha_1, \ldots, \alpha_p, m_q(\alpha_{p+1}, \ldots, \alpha_{p+q}), \alpha_{p+q+1}, \ldots, \alpha_{p+q+r}) = 0 \quad \text{- } A_\infty \text{ relation}$$

or, graphically, $\sum$  $= 0$

where $m_{n+1}: (e_{01})^{\otimes k} \otimes e_1 \otimes (e_{01})^{n-k} \mapsto (-1)^k \binom{n}{k} \frac{B_n}{n!} e_{01}$ — Bernoulli number

for $n \geq 2$

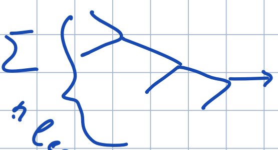
— " — e_0 — " — $\mapsto$ — — " —

anything else $\mapsto 0$

n	0	1	2	3	4	..
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	..

$\sum_{n \geq 0} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}$ - gen. function

• Operations m_n can be constructed (rather than guessed) as sums over trees $\sim$ Feynman graphs for certain (BV) integral

$m_n = \sum$ 
n leaves

A_∞ relations $\sim$ BV quantum master equation.

BF theory. fix M - closed D-manifold, $\mathfrak{g}$ - Lie algebra "BF"

$$S_{cl}(A,B) = \int_M \langle B, \underbrace{dA + \tfrac{1}{2}[A,A]}_{F_A} \rangle = \int_M \langle B, F_A \rangle$$

cl. fields: $\begin{cases} A \in \Omega^1(M,\mathfrak{g}) \\ B \in \Omega^{D-2}(M,\mathfrak{g}^*) \end{cases}$ $\rightsquigarrow$ can generalize A - connection in $\begin{matrix} P \circlearrowleft G \\ \downarrow \\ M \end{matrix}$, $B \in \Omega^{D-2}(M, \mathrm{ad}^*(P))$

e.o.m.: $F_A = 0$, $d_A B = 0$

gauge symmetry: ① $A \to gAg^{-1} + g\,dg^{-1}$, $B \to gBg^{-1}$ ② $A \to A$, $B \to B + d_A\tau$, $\tau \in \Omega^{D-3}(M,\mathfrak{g}^*)$

BV version:
BV (super-) fields

$$\begin{array}{rccccccccl} & 0 & 1 & 2 & 3 & 4 & & D & \leftarrow \text{de Rham degree} \\ \mathcal{A} = & c + & A + & B^+ + & \tau_1^+ + & \tau_2^+ + & \ldots + & \tau_{D-2}^+ & \in \Omega^\bullet(M,\mathfrak{g}) \\ & 1 & 0 & -1 & -2 & -3 & & 1-D & \leftarrow \text{ghost number} \end{array}$$

$$\begin{array}{rccccccl} & 0 & & D-4 & D-3 & D-2 & D-1 & D \\ \mathcal{B} = & \tau_{D-2} + & \ldots + & \tau_2 + & \tau_1 + & B + & A^+ + & c^+ \quad \in \Omega^\bullet(M,\mathfrak{g}^*) \\ & D-2 & & 2 & 1 & 0 & -1 & -2 \leftarrow \text{ghost number} \end{array}$$

two non-homogeneous diff. forms

BV 2-form (odd-symplectic form): $\omega = \int_M \langle \delta\mathcal{B} \overset{\wedge}{,} \delta\mathcal{A} \rangle = \sum_{\Phi \in \{c,A,B,\tau_1,\ldots,\tau_{D-2}\}} \int_M \langle \delta\Phi \overset{\wedge}{,} \delta\Phi^+ \rangle$

BV action: $S = \int_M \langle \mathcal{B}, d\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}] \rangle$ - example of the AKSZ construction with $\mathcal{F} = \mathrm{Map}(T[1]M, \overbrace{\mathfrak{g}[1] \oplus \mathfrak{g}^*[D-2]}^{\mathcal{N}})$

$\Theta = \tfrac{1}{2}\langle p, [c,c] \rangle$ target Hamiltonian

It satisfies the class. master equation

$\{S,S\} = 0$ odd Poisson bracket assoc. to ω

and (with a regularization!) QME:

$$\tfrac{1}{2}\{S,S\} - i\hbar \underbrace{\Delta}_{\int \frac{\delta^2}{\delta\mathcal{A}\,\delta\mathcal{B}}} S = 0.$$

$$\begin{array}{ccc} \mathcal{F} \times T[1]M & \xrightarrow{ev} & \mathcal{N} \\ {\scriptstyle p}\downarrow & & \\ \mathcal{F} & & \end{array}$$

$\omega = \overbrace{p_* ev^*}^{\pi}(\omega_{\mathcal{N}})$

$S = \iota_{d_M^{\text{lifted}}} \pi(\underbrace{\alpha_{\mathcal{N}}}_{\langle p, \delta c \rangle}) + \pi(\Theta)$

$Q_{\mathcal{F}} = d_M^{\text{lifted}} + Q_{\mathcal{N}}^{\text{lifted}}$

Idea: We want to think of BF theory as "associated to" the de Rham algebra $\Omega^\bullet(M)$. Then, construct the "low-energy effective field theory" on cochains of a triangulation $\rightsquigarrow$ read off the algebra on cochains.

Abstract BF theory

Let $V^\bullet, d, [,]$ be a dg Lie algebra s.t. $\mathrm{Str}_g [x,-]=0$ (unimodularity)

Let $\{e_a\}$ - basis in V, $\{e^a\}$ - dual basis

$\rightsquigarrow$ construct the BV package ("abstract BV theory")

fields: $\mathcal{F} = V[1] \oplus V^*[-2]$ $\qquad$ $A = A^a e_a$, $\quad B = B_a e^a$ - superfields

$\omega = \langle \delta B \overset{\wedge}{,} \delta A \rangle$ $\qquad$ (A^a: coords on $V[1]$; B_a: coords on $V^*[-2]$)

$S = \langle B, dA + \frac{1}{2}[A,A] \rangle$ $\leftarrow$ polynomial on $\mathcal{F}$ with coeffs given by structure constants of the algebra

$B_a d^a_b A^b + \frac{1}{2} f^a_{bc} B_a A^b A^c$

CME: $\{S,S\}=0 \iff$
- $d^2=0$
- Leibniz
- Jacobi

QME: —"—, $\Delta S = 0 \iff$ —"—, unimodularity $f^a_{ab}=0$

(1→2)

Effective BV action (for abstract BF theory)

Let (W,d) - deformation retract of (V,d), i.e., $\quad V \rightsquigarrow W$, $(i,p,K) \leftarrow$ "induction data"

$$\begin{tikzcd} V \arrow[loop right, "K"] \arrow[d, shift left, "p"] \\ W \arrow[u, shift left, "i"] \end{tikzcd}$$

i, p chain maps, $p \circ i = \mathrm{id}_W$

$K: V^\bullet \to V^{\bullet-1}$ chain homotopy (contraction): $\quad dK + Kd = \mathrm{id} - i \circ p$

$Ki = pK = 0$

$K^2 = 0$

Rem: space of (i,p,K) triples for V, W fin. dim., inducing a given iso $H^\bullet(V) \underset{i_*}{\overset{p_*}{\rightleftarrows}} H^\bullet(W)$, is contractible.

Then, we have

$V = i(W) \oplus \underbrace{\ker p}_{V'' \text{ acyclic}} = i(W) \oplus V''_{d\text{-ex}} \oplus V''_{K\text{-ex}}$ - "Hodge decomposition"

Ex: $V = \Omega^\bullet(M)$ (Riemannian), $W = H^\bullet(M)$, $i(W) = \Omega_{harm}(M)$

$K = d^*(\Delta + P_{harm})^{-1} \to$ usual Hodge decomp.

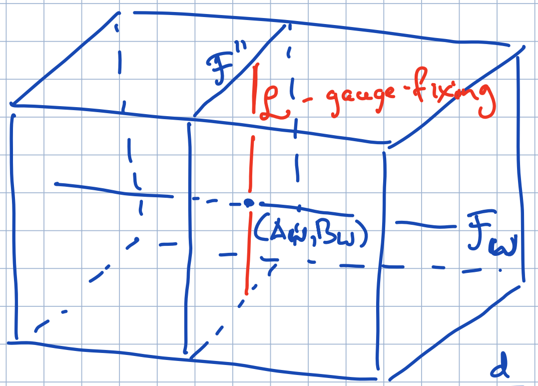
$\mathcal{F}_V = \underbrace{\mathcal{F}_W}_{\substack{\text{slow}\\ \text{fields}}} \oplus \underbrace{\mathcal{F}''}_{\substack{\text{fluctuations /}\\ \text{fast fields}}}$

Define the effective (BV) action S_W on $\mathcal{F}_W$ via

$$(*)\quad e^{\frac{i}{\hbar} S_W(A_W, B_W; \hbar)} := \int_{(A'',B'')\in \mathcal{L}\subset \mathcal{F}''} e^{\frac{i}{\hbar} S_V(A_W + A'', B_W + B'')}$$

$\mathcal{L}$ – Lagrangian

S_V – abstr. BF action assoc. to dgLa structure on V



From Stokes' thm for fiber BV integrals:

① $\Delta_W e^{\frac{i}{\hbar} S_W} = 0 \iff S_W$ satisfies QME

② For $\mathcal{L}_t$ a smooth family of Lagrangians,

$$\frac{d}{dt} S_W^t = \{S_W^t, R^t\} - i\hbar \Delta R^t \text{ for certain } R^t$$

$\left(\sim e^{\frac{i}{\hbar} S_W^t} \text{ for different } t \text{ differ by } \Delta_W(\dots)\right)$

ASIDE

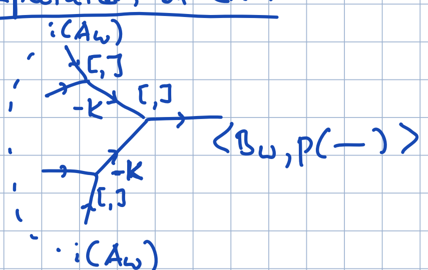
Stokes' thm for fiber BV integrals: let $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$ - product of odd-symp mfds, $\omega = \omega' + \omega''$

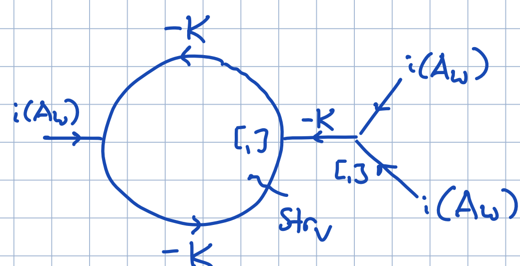
① for $\xi \in \mathrm{Dens}^{1/2}\mathcal{F}$ a half-density, $\boxed{\int_{\mathcal{L}} \Delta \xi = \Delta' \int_{\mathcal{L}} \xi}$

② for $\xi \in \mathrm{Dens}^{1/2}\mathcal{F}$ s.t. $\Delta\xi = 0$ and $\mathcal{L}_t \subset \mathcal{F}''$ a family of Lagr. submfds, $t\in[0,1]$, $\boxed{\int_{\mathcal{L}_1} \xi - \int_{\mathcal{L}_0} \xi = \Delta' \psi}$ for some $\psi \in \mathrm{Dens}^{1/2}\mathcal{F}'$

if $\mathcal{L}_{t+\varepsilon} = \mathrm{Flow}_\varepsilon(X_{H_t})(\mathcal{L}_t)$, $H_t \in C^\infty(\mathcal{L}_t)_{-1}$ then $\psi = \int_0^1 dt \int_{\mathcal{L}_t} \xi H_t$

Feynman diagram computation of $(*)$:

$S_W = \sum_{\Gamma_0 \text{ binary rooted trees}}$ 

$- i\hbar \sum_{\Gamma_1 \text{ 1-loop graphs}}$ 

- Can prove QME for S_W and property ② ($\{K_t\} \to$ can. transf.) via diagrammatics directly!

Vertex: $\rightsquigarrow [,]$

edge: $\rightsquigarrow -K$

leaves: $\rightsquigarrow i(A_W)$

root: $\rightsquigarrow \langle B_W, p(-)\rangle$

QME: $S_W = \underbrace{\langle B_W, dA_W\rangle}_{S_0} + I$; $\quad -\{S_0, I\} = \sum$ graphs w/ one edge marked by $dK + Kd = \mathrm{id} - P_W$

$= \sum$ graphs with one edge marked by P_W $= \underbrace{\frac{1}{2}\{I, I\}}_{\text{marked edge splits the graph}} - i\hbar \Delta_W I$ ← marked edge in the loop

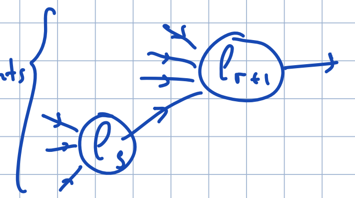
- S_W is generally not itself an abstr. BF theory. However, it has the form

$$S_W = \sum_{n\geq 1} \frac{1}{n!} \langle B_W, l_n^W(A_W,\dots,A_W)\rangle - i\hbar \sum_{n\geq 2} \frac{1}{n!} q_n^W(A_W,\dots,A_W) \quad \text{- "BF}_\infty \text{ theory"}$$

with $l_n : \Lambda^n W \to W$ ($\deg = 2-n$) classical L_∞ operations
$q_n : \Lambda^n W \to \mathbb{R}$ ($\deg = -n$) "quantum operations"

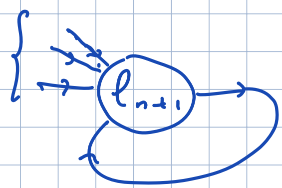
satisfying

(i) $\sum_{r+s=n}$ n inputs



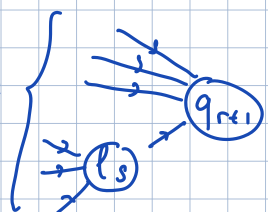
$= 0$

(ii) n



Str_W

$+ \sum_{r+s=n}$ n



$= 0$

"quantum L_∞ algebra" on W

or "unimodular"

- So:

$$\begin{array}{ccc} \text{u. dgLa } V \ (d,[,]) & \longleftrightarrow & \text{BV package } \mathcal{F}_V, \omega_V, S_V \\ \Big\downarrow \text{homotopy transfer formulae} & & \Big\downarrow \text{fiber BV integral} \\ qL_\infty \text{ alg. } W \ \{l_n, q_n\} & \longleftrightarrow & \mathcal{F}_W, \omega_W, S_W \end{array}$$

(refinement of Kontsevich-Soibelman f-la)

de Rham $\to$ triangulation

Whitney / Dupont construction

On $\Delta^n = \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \ \sum_i t_i = 1\}$
standard simplex

Whitney elementary form:

$$\underbrace{\chi_{i_0\cdots i_k}}_{k\text{-face } \sigma\subset\Delta^n} = k! \sum_{j=0}^{k} (-1)^j t_{i_j}\, dt_{i_0} \wedge \cdots \widehat{dt_{i_j}} \cdots \wedge dt_{i_k} \quad (**)$$

Ex: on Δ^1:

$\chi_0 = t_0 = 1-t$ ← $t = t_1$
$\chi_1 = t_1 = t$
$\chi_{01} = dt$

for M - mfd, T - triangulation, $p: \Omega^\bullet(M) \to C^\bullet(T)$, $\alpha \mapsto \sum_\sigma e_\sigma \left(\int_\sigma \alpha\right)$ Poincaré integration map

$i: C^\bullet(T) \to \Omega^\bullet(M)$

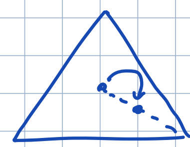
$\sum_\sigma \alpha^\sigma e_\sigma \longmapsto \sum_\sigma \alpha^\sigma \chi_\sigma$ Whitney form assoc. to σ

· On any $\sigma' \supset \sigma$, $\chi_\sigma^{(\sigma')}$ is given by (**).

it is a chain map and satisfies $p \circ i = id$

Dupont's operator

$$\begin{array}{ccc} \Delta^n \times [0,1] & \xrightarrow{\varphi_i} & \Delta^n \\ (t_0,\dots,t_n; u) & \longmapsto & (ut_0, \dots, 1-u+ut_i, \dots, ut_n) \\ \downarrow \pi & & \\ \Delta^n & & \end{array}$$



i-th vertex

φ_i ("homothety") pulls points toward i-th vertex by a factor u.

$h^i = \pi_* \varphi_i^* : \Omega^\bullet(\Delta^n) \to \Omega^{\bullet-1}(\Delta^n)$ - homotopy between id and ev_i

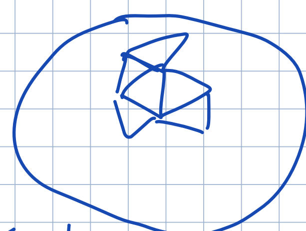
$$K_{\Delta^n} = \sum_{k=0}^{n-1} (-1)^k \sum_{0 \le i_0 < \dots < i_k \le n} \chi_{i_0 \dots i_k} \wedge h^{i_k} \dots h^{i_0}$$

Thm (Whitney-Dupont-Getzler) → $K^2 = 0$

$\Omega^\bullet(\Delta^n) \underset{(i_{\Delta^n}, p_{\Delta^n}, K_{\Delta^n})}{\rightsquigarrow} C^\bullet(\Delta^n)$ is a collection of induction triples compatible with face maps and S_{n+1}-symmetry

glue simplices

$$\Omega^\bullet(M) \underset{(i,p,K)}{\rightsquigarrow} C^\bullet(T)$$



$K\alpha|_\sigma = K^{(\sigma)} \alpha|_\sigma$

Effective action on cell cochains of a triangulation

Set $V = \Omega^\bullet(M) \otimes \mathfrak{g}, d, [,]$ ← Lie alg. of coeffs

$W = C^\bullet(T) \otimes \mathfrak{g}$

Fields: $A_T = \sum_\sigma e_\sigma A^\sigma$ $\in \mathfrak{g}$, gh $= 1-|\sigma|$ - cell cochain

$B_T = \sum_\sigma e^\sigma B_\sigma$ - chain

Form of the answer: ("locality")

$$S_T = \sum_\sigma \bar{S}_\sigma\left(\{A^{\sigma'}\}_{\sigma' \subset \sigma}, B_\sigma\right)$$ building blocks

f-la depends only on dim σ and restrictions of A, B to faces of σ.

Building blocks

generates the A_∞ algebra of the interval we started with.

- $\bar S_{\Delta^0} = \frac12 \langle B_0, \frac12 [A^0, A^0] \rangle$

- $$\bar S_{\Delta^1} = \Big\langle B_{01}, \tfrac12 [A^{01}, A^0 + A^1] + \underbrace{\left(\frac{\mathrm{ad}_{A^{01}}}{2} \coth \frac{\mathrm{ad}_{A^{01}}}{2}\right)}_{\sum_{k\geq 0} \frac{B_{2k}}{(2k)!} (\mathrm{ad}_{A^{01}})^{2k} = F(\mathrm{ad}_{A^{01}})} \circ (A^1 - A^0) \Big\rangle - i\hbar\, \mathrm{tr}_g \log \underbrace{\frac{\sinh \frac{\mathrm{ad}_{A^{01}}}{2}}{\frac{\mathrm{ad}_{A^{01}}}{2}}}_{G(\mathrm{ad}_{A^{01}})} \qquad (\#)$$

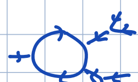
$$F(x) = 1 + \frac{x^2}{12} - \frac{x^4}{720} + \ldots \qquad G(x) = \frac{x^2}{2\cdot 12} - \frac{x^4}{4\cdot 720} + \ldots$$

(@)

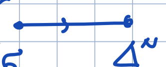
- $$\underset{\substack{\cap \\ \mathrm{Fun}(F_{\Delta^N})[\hbar]}}{\bar S_{\Delta^N}} = \sum_{\substack{\Gamma_0 \text{ binary rooted} \\ \text{trees with} \\ n \geq 1 \text{ leaves}}} \sum_{\substack{\sigma_1, \ldots, \sigma_n \subset \Delta^N \\ \text{sub-simplices} \\ \text{decorating} \\ \text{leaves}}} \frac{1}{|\mathrm{Aut}\,\Gamma_0|} C^{\Delta^N}_{\Gamma_0; \sigma_1 \ldots \sigma_n} \langle B_{\Delta^N}, \underbrace{\mathrm{Jacobi}_{\Gamma_0}(A_{\sigma_1}, \ldots, A_{\sigma_n})}_{\text{nested commutator determined by } \Gamma_0} \rangle -$$

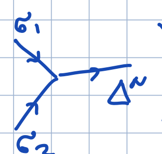
structure constants

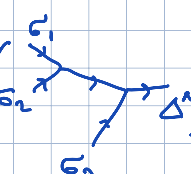
$$- i\hbar \sum_{\substack{\Gamma_1 \text{ 1-loop} \\ \text{with } n \geq 2 \text{ leaves}}} \sum_{\sigma_1, \ldots, \sigma_n \subset \Delta^N} \frac{1}{|\mathrm{Aut}\,\Gamma_1|} C^{\Delta^N}_{\Gamma_1; \sigma_1 \ldots \sigma_n} \mathrm{tr}_g\, \mathrm{Jacobi}_{\Gamma_1^{\mathrm{cut}}}(A_{\sigma_1}, \ldots, A_{\sigma_n})$$

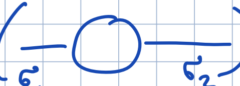


Structure constants: (examples)

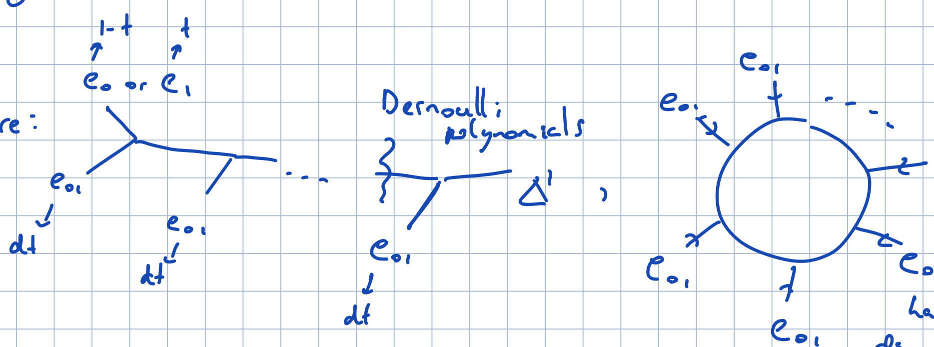
- $C\Big($  $\Big) = \begin{cases} \pm 1 & \text{if } \sigma \text{ face of codim} = 1 \\ 0 & \text{otherwise} \end{cases}$

- $C\Big($  $\Big) = \begin{cases} \pm \dfrac{|\sigma_1|!\,|\sigma_2|!}{(N+1)!} & \text{if } |\sigma_1| + |\sigma_2| = N,\ \sigma_1 \cap \sigma_2 = pt \\ 0 & \text{otherwise} \end{cases}$

- $C\Big($  $\Big) = \begin{cases} \pm \dfrac{|\sigma_1|!\,|\sigma_2|!\,|\sigma_3|!}{(|\sigma_1| + |\sigma_2| + 1) \cdot (N+2)!} \\ 0 \end{cases}$

- $C\Big($  $\Big) = \begin{cases} \pm \dfrac{1}{(N+1)^2 (N+2)} & \text{if } \sigma_1 = \sigma_2,\ |\sigma_1| = 1 \\ 0 & \text{otherwise} \end{cases}$

- In (#) contributing graph are:



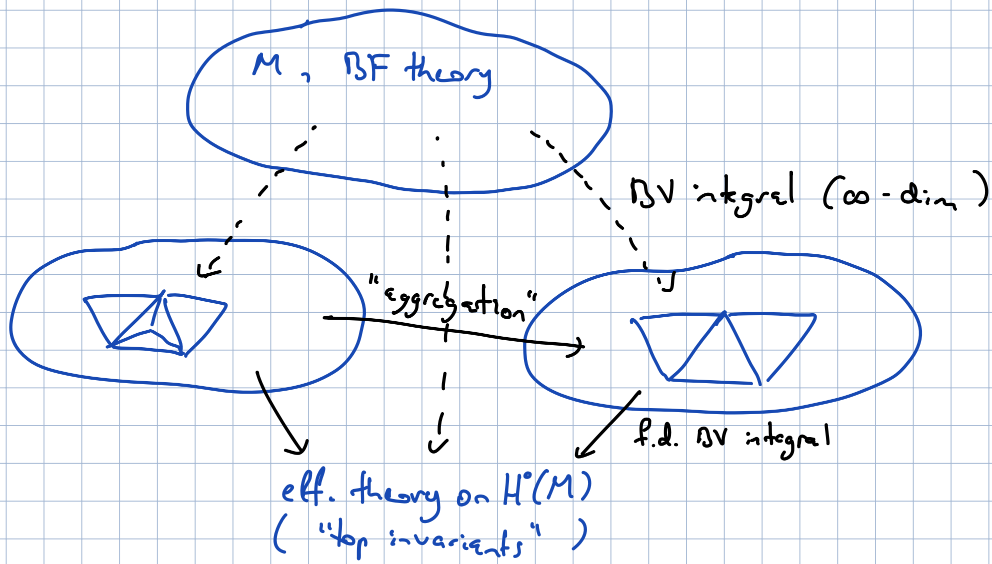
wheel with n spikes in basis $\{t^k\}$, the corresp. operator has only n nonzero diagonal entries! → Str exists

· On Δ^N, $\mathrm{Str}_{\Omega^\bullet(\Delta^N)}$ needs a regularization

Way 1: replace Dupont's op. $K \to K_\varepsilon$, replacing the domain for u with $[0, 1-\varepsilon]$ in Dupont's construction

$\lim_{\varepsilon\to 0}$ in Feynman diagrams is finite!

Way 2: one can recover 1-loop part of $\bar{S}_{\Delta^N}$ from $\bar{S}^{(0)}_{\Delta^N}$ from QME



$C^\bullet(T) \rightsquigarrow H^\bullet(M)$ $\quad$ $S_{H^\bullet}$ is a BF_∞-type effective theory - a gen. fun. for qL_∞ algebra str. on $H^\bullet(M,\mathfrak{g})$

class. L_∞ operations $l_n^{H^\bullet}$ = Massey-Lie operations - remember $\pi_k(M)\otimes\mathbb{Q}$ if $\pi_1(M)=0$.

quantum operations = stronger invariant

Ex:

$S_{H^\bullet(S^1)}$	$S_{H^\bullet(\text{Klein bottle})}$
$\langle B_0, \frac{1}{2}[A^0,A^0]\rangle + \langle B_1, [A^1, A^0]\rangle -$	—— " ——
$-i\hbar\,\mathrm{tr}_{\mathfrak{g}} \log \boxed{\dfrac{\sinh \frac{\mathrm{ad}_{A^1}}{2}}{\frac{\mathrm{ad}_{A^1}}{2}}}$	$-i\hbar\,\mathrm{tr}_{\mathfrak{g}} \log \boxed{\left(\frac{\mathrm{ad}_{A^1}}{2} \coth \frac{\mathrm{ad}_{A^1}}{2}\right)^{-1}}$

← different quantum operations!

Mathematical formulation of the result:

Thm: $\exists$ values $C^{\Delta^N}_{\Gamma_{0,1};\sigma_1\ldots\sigma_n}$ of str. const. s.t. for any simplicial cx T, $S_T := \sum_{\sigma\in T} \bar{S}_\sigma$ satisfies QME.

defined by ansatz (@)

Proof: • on Δ^N, $S^{(0)}_{\Delta^N} = \sum_{\text{trees}}$ is well-defined, satisfies CME (diagrammatic proof) • on Δ^N, can find $S^{(1)}$ s.t. $S^{(0)} + \hbar S^{(1)}$ solve QME (uniquely up to $Q(\ldots)$)

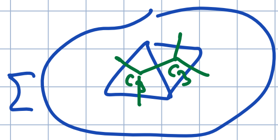
• inclusion-exclusion: $X = X_1 \cup_Y X_2$ simpl cx, $S_{X_{1,2}}, S_Y$ satisfy QME and BF_∞ ansatz $\Rightarrow S_X = S_{X_1} + S_{X_2} - S_Y$ satisfies QME.

Combinatorial 2d TCFT from local cyclic A_∞ algebra

(j.w. with J. Beck, A. Losev, arXiv: 2402.04468)

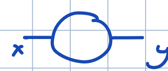
- warm-up: fix (V, m_2, g) an assoc algebra with pairing, $C_3 = g(-, m_2(-,-))$

(Σ, T) triangulated surface $\longrightarrow Z(\Sigma,T) = \langle \bigotimes_{\text{2-simplices}} C_3 , \bigotimes_{\text{edges}} g^{-1} \rangle \in \mathbb{k}$



Invariance w.r.t. Pachner moves:

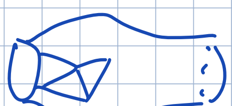
$\sim$ associativity of m_2 — true if $g(x,y) = \mathrm{tr}\, m_2(x, m_2(y,-))$



- Extends to a TQFT functor

$\mathrm{Cob}_2^{\text{triang}} \longrightarrow \mathrm{Vect}$

$S^1_{(k)} \longmapsto V^{\otimes k}$



$\longmapsto (Z(\Sigma,T) : \mathcal{H}_{in} \to \mathcal{H}_{out})$

- Factors through a reduced theory indep. of triang., $\mathrm{Cob}_2 \to \mathrm{Vect}$

$S^1 \longmapsto \mathcal{H}^{red} = \mathrm{Center}(V)$

- for $V = \mathbb{C}[G]$ ← fin. grp, $Z(\text{genus } h) = \sum_{R \in \mathrm{irrep}(G)} \left(\frac{\dim R}{|G|}\right)^{2-2h} = \mathrm{Vol}_{\mathrm{gpd}}\, \mathcal{M}^{\text{coverings}}_{\Sigma, G}$

Idea: relax invariance wrt $\text{Pachner}_{I,II}$ up to homotopy: $\boxed{\square} - \boxed{\square} = Q(\square)$, $\triangle - \triangle = Q(\triangle)$ "stress-energy tensor"

HTQFT setup: Segal's QFT: $Z(\Sigma) \in \mathrm{Fun}(\mathrm{Geom}_\Sigma) \otimes \mathrm{Hom}(\mathcal{H}_{in}, \mathcal{H}_{out})$

HTQFT: $Z(\Sigma) \in \boxed{\Omega^\bullet}(\mathrm{Geom}_\Sigma) \otimes$ —"—

s.t. $\boxed{(d_{\mathrm{Geom}} + Q) Z = 0}$ $\rightsquigarrow$

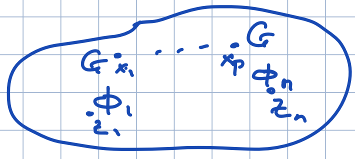
← form degree on Geom

$\rightsquigarrow Q Z^{(0)} = 0$

$d_{\mathrm{Geom}} Z^{(0)} = -Q Z^{(1)}$

$\dots$

Ex: • 2d TCFT : $T = Q(G)$

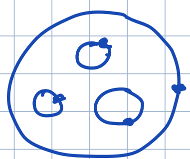


$$\langle \underbrace{G \cdots G}_{p}\, \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle \rightsquigarrow$$

$$Z^{(p)} \in \Omega^p(\widetilde{\mathcal{M}}_{h,n}, \mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathbb{C}))$$

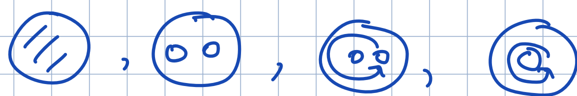
satisfies $(d_{\mathcal{M}} + Q) Z = 0$

• <u>2d TCFT in genus zero</u> $\sim$ algebra over the operad E_2^{fr}



$$Z^{(p)}_{[n]} \in \Omega^p(E_2^{fr}(n), \mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathcal{H}))$$

satisfying $(d_{E_2^{fr}} + Q) Z = 0.$

$H_\bullet(E_2^{fr})$ is generated by 

BV-cycle

• $H_Q(\mathcal{H})$ has an action of $H_\bullet(E_2^{fr})$

i.e. is a BV algebra. $Z($  $) = G_{0,-}$ BV operator.

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Towards combTCFT: $\Sigma \rightsquigarrow \Xi$ CW cx of triangulations (Pachner complex)

• vertices of Ξ = triangulations

• edges = Pachner moves

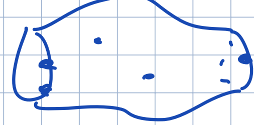
$$Z(\Sigma) \in C^\bullet(\Xi) \otimes \mathrm{Hom}(\mathcal{H}_{in}, \mathcal{H}_{out})$$

s.t. $(\delta_\Xi + Q) Z = 0 \rightsquigarrow Q Z^{(0)} = 0, \quad \delta_\Xi Z^{(0)} = -Q Z^{(1)}$ - Pachner moves change $Z^{(0)}$ by a Q-exact term.

Flip theory: Fix $(V, Q = m_1, m_2, m_3, \dots ; g)$ - cyclic A_∞ algebra

$c_{n+1} = g(-, m_n(-, \dots, -))$ cyclic op

Fix Σ, P - set of vertices



$\Xi_{flip}(\Sigma, P)$ - CW cx with

• 0-cells $\sim$ triangs of Σ w/ vertices in P

• k-cell e_α $\sim$ polygonal decomp$^\alpha$ of Σ with $\sum_{\text{2-cells } p} (|p| - 3) = k$

• boundary of e_α $\sim$ subdivisions of one p by a diagonal 

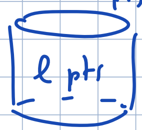
• $e_\alpha \underset{\text{combinatorially equiv}}{\sim} \prod_p K_{|p|-1}$ ← Stasheff's associahedron
~~~

cTCFT: $Z \in C^\bullet(\Xi_{flip}) \otimes \mathrm{Hom}(\mathcal{H}_{in}, \mathcal{H}_{out})$

with $Z(e_\alpha) = \langle \bigotimes_{2\text{-cells } p} C_{|p|}, \bigotimes_{\text{edges}} g^{-1} \rangle$

<u>Thm:</u> (1) Z defines a functor from $Cob_2^{\text{polygonal decomp}} \to Vect$

(2) $(\delta_{\Xi_{flip}} + Q) Z = 0$

<u>Comb. BV operators:</u> $c \in C^\bullet(\Xi_{flip}($ 

$))$ is a "BV cycle" if

- $\partial c = 0$
- $c \circ c = \partial(\underbrace{\cdots}_{g\text{-chain}})$

then $Z(c): \mathcal{H}_{S^1_{(k)}} \to \mathcal{H}_{S^1_{(k)}}$ - combinatorial BV operator: $Q\, Z(c) = 0$
$Z(c)^2 = Q(\cdots)$.

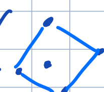
"Secondary polytope theory"

$A \subset \mathbb{R}^2$ in gen. pos. $\rightsquigarrow \Xi_{sp}(P,A) \subset \mathbb{R}^{|A|-3}$ convex polytope ("secondary polytope")

$P := \mathrm{conv}(A)$ vertices $\sim$ regular triangulations of P w/ vertices at A

faces $\sim$ "marked polygonal subdivisions" α of (P, A)

algebraic input - "$\hat{A}_\infty$ algebra"

V, Q, $c($  $): V^{\otimes n} \to \mathbb{k}$, s.t. $Q(m_{[A]}) = \sum_{\substack{\text{coarse subdivisions} \\ \text{of } (P,A)}} \langle \bigotimes c_{[A_i]}, \bigotimes g^{-1} \rangle$

"config chamber"

Define $Z \in C^\bullet(\Xi_{sp}) \otimes \underbrace{V^{\otimes n}}_{\mathcal{H}_{out}}$, $Z(e_\alpha) = \langle \bigotimes_{2\text{-cells}} c_{[A_i]}, \bigotimes_{\text{edges}} g^{-1} \rangle$ $(*)$

- satisfies $(\delta_\Xi + Q) Z = 0$.

<u>"Dream theory"</u>: fix Σ surface. Conjecture: $\exists$ CW complex Ξ s.t. · vertices $\sim$ triangulations

· edges $\sim$ Pachner moves · $\Xi \underset{\text{homotopy equiv.}}{\sim} \widetilde{\mathcal{M}}_{h,n}$ · cells of $\Xi \sim$ polygonal decomp. w/ floating points

Then for an $\hat{A}_\infty$ algebra V, one can define an "ideal cTCFT" by $(*)$.

Dictionary:

Comb TCFT	continuum TCFT
triangulated surface	surface w/ metric
	deformation of cx str.
	Weyl transform of metric
Pachner complex Ξ —"— for $h=0$	$\tilde{\mathcal{M}}_{h,n}$ E_2^{fr}
$Z \in C^{\cdot}(\Xi) \otimes \mathrm{Hom}(\mathcal{H}_{in}, \mathcal{H}_{out})$	$Z \in \Omega^{\cdot}(\tilde{\mathcal{M}}_{h,n}) \otimes$ —"—
$(\delta_{\Xi} + Q) Z = 0$	$(d + Q) Z = 0$
state on a polygon	quantum field / point observable
Z paired with states in polygonal holes	correlator of observables
A_∞ op. C_4	field $G^{\text{traceless}}$
$\hat{A}_\infty$ op $c(\triangle)$	field $\mathrm{tr}\, G$