2D Yang-Mills on surfaces with corners in BV formalism

with Riccardo Iraso

* want to recover Migdal's answer $Z(\Sigma) = \sum_{R\text{-irrep of } G} (\dim R)^{2-2h} e^{\frac{i}{\hbar} a\, C_2(R)}$, $a = \mathrm{area}(\Sigma)$
  from perturbative path integral
* example of BV-BFV quantization programme (BV on mfld with boundary, satisfying cutting-pasting)
  ref: arXiv: 1507.01221

2D Yang-Mills on a closed surface Σ.

G - Lie group (compact, simple, simply-connected)

$$S_{YM}(A,B) = \int_\Sigma \langle B, dA + \tfrac{1}{2}[A,A]\rangle + \tfrac{1}{2}\mu(B,B)$$

$A \in \mathrm{Conn}_{\Sigma,G}$, $B \in \mathrm{Fun}(\Sigma, \mathfrak{g}^*)$, μ - area form

gauge symmetry $(A,B) \sim (A^g, B^g)$, $A^g = g^{-1}Ag + g^{-1}dg$, $B^g = g^{-1}Bg$

BV:
$$S(\mathcal{A},\mathcal{B}) = \int_\Sigma \langle \mathcal{B}, d\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}]\rangle + \tfrac{1}{2}\mu(\mathcal{B},\mathcal{B})$$

$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{A}^{(2)} = c + A + B^+$ $\qquad \mathcal{F} = \Omega^\bullet(\Sigma,\mathfrak{g})[1] \oplus \Omega^\bullet(\Sigma,\mathfrak{g}^*)$

$\mathcal{B} = \mathcal{B}^{(0)} + \mathcal{B}^{(1)} + \mathcal{B}^{(2)} = B + A^+ + c^+$ $\qquad \omega = \int_\Sigma \langle \delta\mathcal{B}, \delta\mathcal{A}\rangle$ - -1- symplectic 2-form on $\mathcal{F}$.

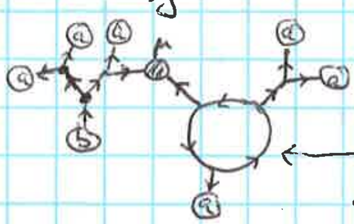
CME: $\{S,S\} = 0$, QME: $\frac{1}{2}\{S,S\} - i\hbar\Delta S = 0 \iff \Delta e^{\frac{i}{\hbar}S} = 0$, $\Delta = \int_\Sigma \frac{\delta}{\delta\mathcal{A}}\frac{\delta}{\delta\mathcal{B}}$

Perturbative PI Quantization:

$\mathcal{F} = \mathcal{F}_{res} \oplus \mathcal{F}_{fluct}$, $(a,b) \in \mathcal{F}_{res}$, fast fields in $\mathcal{F}_{fluct}$

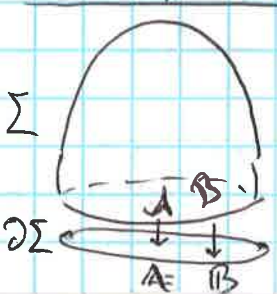
$$Z(a,b) = e^{\frac{i}{\hbar}S_{eff}(a,b)} = \int_{\mathcal{L}\subset\mathcal{F}_{fluct}} e^{\frac{i}{\hbar}S(a+\mathcal{A}_{fl},\, b+\mathcal{B}_{fl})}\, D\mathcal{A}_{fl}\, D\mathcal{B}_{fl}$$

Smallest model: $\mathcal{F}_{res} = H^\bullet(\Sigma,\mathfrak{g})[1] \oplus H^\bullet(\Sigma,\mathfrak{g}^*)$ (zero modes)

$S_{eff}(a,b) = \sum$ 

propagator $\eta \in \Omega^1(\mathrm{Conf}_2(\Sigma))$,
s.t. $K = \int_2 \eta_{12}(\cdots)_2 : \Omega^\bullet(\Sigma) \to \Omega^{\bullet-1}(\Sigma)$
- chain homotopy between $\mathbb{1}$ and $P_{H^\bullet}$

Classical BV-BFV picture on surfaces with boundary



$\Sigma \to \mathcal{F}, \omega, Q, S$ (ω - -1-symp, Q - hom. v.f. / BRST-operator)

$\downarrow \pi = \iota^*$, π_*

$\partial\Sigma \to \Phi, \omega_\partial = \delta\alpha_\partial, Q_\partial, S_\partial$ (ω_∂ 0-symplectic, S_∂ gh=1 BFV action)

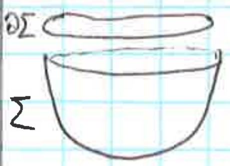
$$\boxed{\iota_Q\omega = \delta S + \pi^*\alpha_\partial}$$

$\Phi = \Omega^\bullet(\partial\Sigma,\mathfrak{g})[1] \oplus \Omega^\bullet(\partial\Sigma,\mathfrak{g}^*) \ni (\mathbb{A},\mathbb{B})$

$\alpha_\partial = \int_\partial \langle \mathbb{B}, \delta\mathbb{A}\rangle$ $\qquad \omega_\partial = \int_\partial \langle \delta\mathbb{B}, \delta\mathbb{A}\rangle$

$S_\partial = \int_\partial \langle \mathbb{B}, d\mathbb{A} + \tfrac{1}{2}[\mathbb{A},\mathbb{A}]\rangle$

Quantum picture



$\longrightarrow$ space of states $\mathcal{H}^{P}_{\partial}$ (polarization data), Ω_∂ - BFV differential ; $\Omega_\partial^2 = 0$

$\longrightarrow \mathcal{F}_{res}, \omega_{res}$ — -1-sympl. structure; $\mathcal{F}_{res}$: space of residual fields

$Z_\Sigma \in \mathcal{H}_{\partial\Sigma} \otimes \mathrm{Fun}(\mathcal{F}_{res})$; $\boxed{(\hbar^2 \Delta_{res} + \Omega_\partial) Z_\Sigma = 0}$ mQME

quantization

$\Phi, \omega_\partial \rightsquigarrow \mathcal{H}_\partial = \mathrm{Fun}(\{\mathbb{P}\})$, $\Omega_\partial = \widehat{S_\partial}$ — geom. quantization

Lagr. polarization: $p \downarrow$ $\{\mathbb{P}\}$ ← base fields (boundary conditions)

$$\begin{array}{c} \mathcal{F} \\ \pi \downarrow \\ \Phi \\ p \downarrow \\ \{\mathbb{P}\} \end{array}$$

$\mathbb{P} \in \{\mathbb{A}, \mathbb{B}\}$

for a given $\mathbb{P}$, $\mathcal{F}_{\mathbb{P}} = \pi^{-1} p^{-1} \mathbb{P} = \mathcal{F}_{res} \oplus \mathcal{F}_{fluct}$, where $\mathcal{F}_{fluct} \leftarrow \Omega^\cdot(\Sigma, \partial_{\mathbb{A}})[1] \oplus \Omega^\cdot(\Sigma, \partial_{\mathbb{B}})$

$$Z(\underbrace{\mathbb{P}}_{\mathbb{A} \text{ or } \mathbb{B} \text{ on bdry}}; \underbrace{a, b}_{\text{res. fields}}) = \int_{\mathcal{F}_{fluct}} e^{\frac{i}{\hbar} S(\mathbb{A}|_{\partial_{\mathbb{A}}} + a + A_{fl},\, \mathbb{B}|_{\partial_{\mathbb{B}}} + b + B_{fl})}$$

Spaces of states (for a circle)

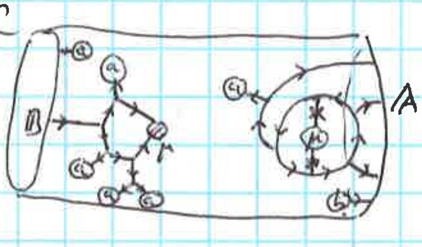
$\mathcal{H}^{\mathbb{A}}_{S^1} = \mathrm{Fun}(\Omega^\cdot(S^1, \mathfrak{g})[1])$, $\Omega^{\mathbb{A}}_{S^1} = i\hbar \int_{S^1} \langle d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}], \frac{\delta}{\delta \mathbb{A}} \rangle$

Note: $H^0_{\Omega^{\mathbb{A}}_{S^1}} = \mathrm{Fun}(G)^G$ ← class functions on G.

$\mathcal{H}^{\mathbb{B}}_{S^1} = \mathrm{Fun}(\Omega^\cdot(S^1, \mathfrak{g}^*))$, $\Omega^{\mathbb{B}}_{S^1} = \int_{S^1} \langle d\mathbb{B}, i\hbar \frac{\delta}{\delta \mathbb{B}} \rangle + \frac{(i\hbar)^2}{2} \langle \mathbb{B}, [\frac{\delta}{\delta \mathbb{B}}, \frac{\delta}{\delta \mathbb{B}}] \rangle$

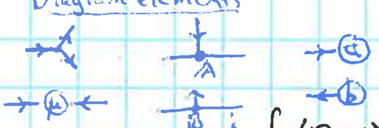
Partition function

$Z = \sum$

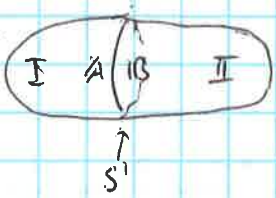


propagator $\eta \sim K : \Omega^\cdot(\Sigma, \partial_{\mathbb{A}}) \to \Omega^{\cdot - 1}(\Sigma, \partial_{\mathbb{A}})$
- chain contraction $\Omega^\cdot(\Sigma, \partial_{\mathbb{A}}) \rightsquigarrow H^\cdot(\Sigma, \partial_{\mathbb{A}})$

Diagram elements



Gluing:



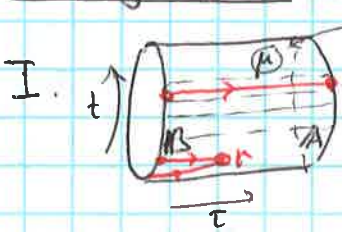
$$Z = \int_{\mathbb{A}, \mathbb{B}} Z_{\mathrm{I}}(\cdots, \mathbb{A}) e^{-\frac{i}{\hbar}\int_{S^1} \langle \mathbb{B}, \mathbb{A} \rangle} Z_{\mathrm{II}}(\mathbb{B}, \cdots)$$

"Segal's gluing"

- corresponds to using a "glued propagator" on Σ, "$\eta_\Sigma = \eta_{\mathrm{I}} * \eta_{\mathrm{II}}$"

Building blocks for 2D YM

I.

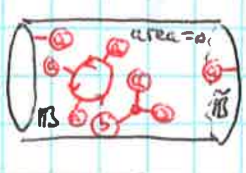


nonzero area form allowed

axial gauge: $\eta(t,\tau \to t',\tau') = -\Theta(\tau' - \tau)\,\delta(t - t')(dt' - dt)$

$$Z = e^{\frac{i}{\hbar} \oint \langle \mathbb{B}, \mathbb{A} \rangle - \frac{1}{2} P_* \mu (\mathbb{B}, \mathbb{B})}$$

II.



$a = a_0 \cdot 1 + a_1\, dt$

$b = b^1\, d\tau + b^0\, dt\, d\tau$

$$Z = e^{\frac{i}{\hbar}\left(\frac{1}{2}\int_\Sigma \langle b, [a,a] \rangle + \int_{S^1} \langle \mathbb{B} - \tilde{\mathbb{B}}, a \rangle\right)} \boxed{\det_{\mathfrak{g}} \frac{\sinh \frac{ad_{a_1}}{2}}{\frac{ad_{a_1}}{2}}}$$

$e^{\sum \text{wheels}}$

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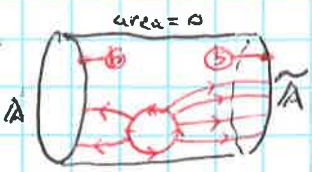
05/25/2015
(3)

III. area = 0



"globalization"

holonomy

$$\int_{\mathcal{L}\subset\mathcal{F}_{res}} Z = \delta_{\mathfrak{g}}\big(\log U(\mathbb{A}),\ \log U(\tilde{\mathbb{A}})\big)\cdot\Bigg(\det \underbrace{\frac{\sinh\frac{\mathrm{ad}_{\log U(\mathbb{A})}}{2}}{\frac{\mathrm{ad}_{\log U(\mathbb{A})}}{2}}}_{\text{Jacobian of } \exp:\mathfrak{g}\to G}\Bigg)^{-1}$$

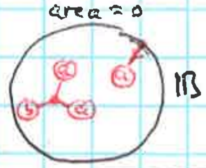
$$\int_{\mathcal{L}\subset\mathcal{F}_{res}}\big(\cdot\,\delta_{\Omega_\partial}(\mathbb{A}^{(0)}_p,\tilde{\mathbb{A}}^{(0)}_{\tilde{p}})\big)$$

$a=0$, b any

$$= \delta_G\big(U(\mathbb{A}),\ U(\tilde{\mathbb{A}})\big)$$

← this answer is recovered from the case $\mathbb{A}=\mathbb{A}\,dt$ - constant 1-form, by using gauge-symmetry / mQME on Z.

IV. area = 0



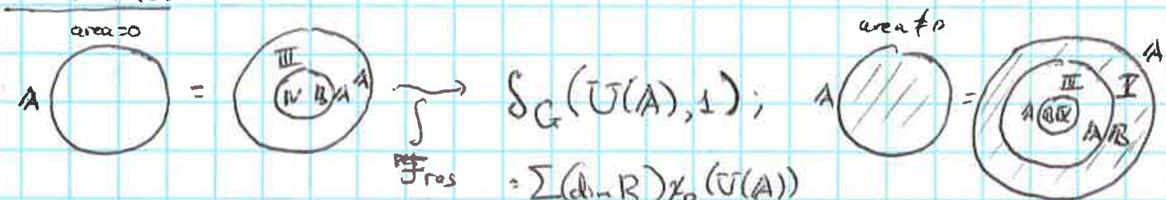
any gauge-fixing $a = a_0\cdot 1$ $b = b^0\, d\,\mathrm{vol}$

$$Z = e^{\frac{i}{\hbar}\left(-\int_{\partial D}\langle\mathbb{B},a\rangle+\frac{1}{2}\int_D\langle b,[a,a]\rangle\right)}$$

Glued surfaces:

$\hat{H}_{YM}$ - comes from Fourier transform of Z_{I}

area = 0   area ≠ 0

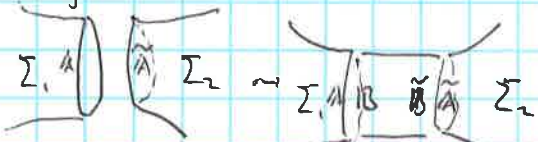


$$\xrightarrow{\int_{\mathcal{F}_{res}}} \delta_G(U(\mathbb{A}),1);\qquad = e^{-\frac{i}{2\hbar}a\left(\frac{\delta}{\delta\mathbb{A}},\frac{\delta}{\delta\mathbb{A}}\right)}\delta(U(\mathbb{A}),1)$$

$$=\sum_R(\dim R)\,\chi_R(U(\mathbb{A}))\qquad =\sum_R(\dim R)\,e^{-\frac{ia}{\hbar}C_2(R)}\chi_R(U(\mathbb{A}))$$

character

Gluing $\mathbb{A}$-circles:



$$Z_\Sigma = \int_G dU\ Z_{\Sigma_1}(\cdots,\underset{U(\mathbb{A})}{U})\,Z_{\Sigma_2}(\underset{U(\tilde{\mathbb{A}})}{U},\cdots)$$

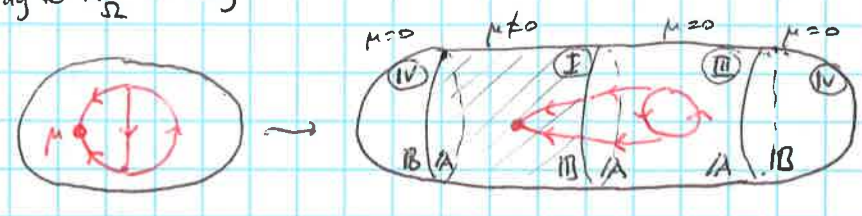
[should be globalized]

• For Σ possibly with bdry in $\mathbb{A}$-pol., $\chi(\Sigma)\geq 0$,

$$\Bigg[\int_{\mathcal{L}\subset\mathcal{F}_{res}}^{\text{BV-BFV}} Z(\Sigma)\Bigg]_{\text{passing to } H^0_{\Omega} \text{ on bdry}} = \sum_R(\dim R)^{\chi(\Sigma)}e^{\frac{i}{\hbar}a\frac{1}{2}C_2(R)}\prod_{k=1}^{n}\chi_R(U_k)\qquad (*)$$

n = # bdry components

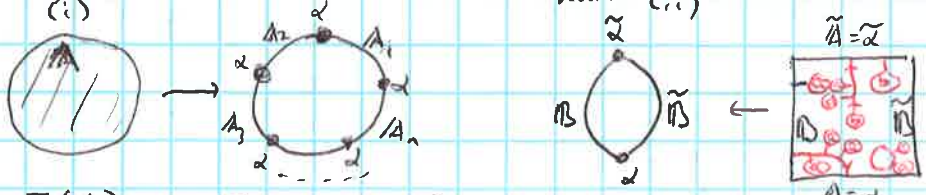
• complicated YM diagrams become computable in a special gauge



Surfaces with $\chi<0$ and corners

Intervals decorated with $\mathbb{A},\mathbb{B}$ (polarization data); corners with α,β - fixing corner value of field A or B. Allowed gluing:

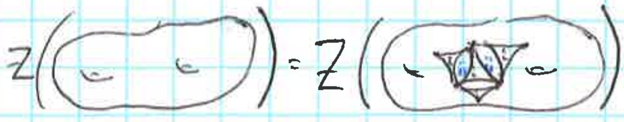
(i)   "bean" (ii)



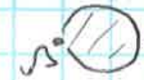
$Z = \cdots$ - uses gen. functions for Bernoulli numbers & polynomials

$Z(\mathbb{A}) \longrightarrow Z(\mathbb{A}_1 * \cdots * \mathbb{A}_n)$

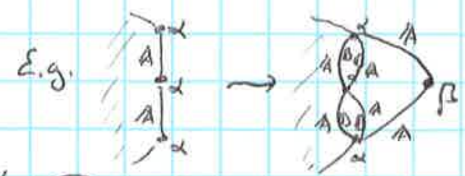
concatenation



- satisfies comparison THM (*) !

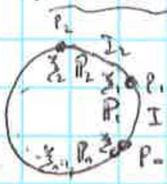
Using additional building blocks  , 

$Z = e^{-\frac{i}{\hbar}\langle\int_{\mathbb{B}} \mathbb{B}, \alpha\rangle}$ $\quad \to Z = e^{\frac{i}{\hbar}\langle \beta, \log U(\mathbb{A})\rangle}$

can glue any surface with combination of bdry / corners in any polarizations. E.g. 

Gluing:
$$\int_{\mathbb{A},\mathbb{B}} Z\left(\left(\Sigma_1 \;\; \mathbb{A}\right|\right) e^{-\frac{i}{\hbar}\int\langle\mathbb{B},\mathbb{A}\rangle} Z\left(\left|\mathbb{B}\;\Sigma_2\right)\right) = Z\left(\left(\Sigma_1 \,\vdots\, \Sigma_2\right)\right)$$

Space of states, Ω and mQME.



$\mathcal{H} = \{\Psi(\xi_0, \mathbb{P}_1, \xi_1, \ldots, \xi_{n-1}, \mathbb{P}_n)\}$

if ξ_k agrees with $\mathbb{P}_k$, restriction of $\mathbb{P}_k$ to p_k is ξ_k.

corner-edge terms, pure corner Ω

$$\Omega = \sum_k \Omega_{I_k}^{\mathbb{P}_k} + \underbrace{\sum\left(\Omega_{p_k}^{\mathbb{P}_k\xi_k} + \Omega_{p_k}^{\xi_k} + \Omega_{p_k}^{\xi_k\mathbb{P}_{k+1}}\right)}_{\text{corner contribution}}$$

$\mathbb{A}$ or $\mathbb{B}$

$\Omega_p^{\alpha} = \frac{i\hbar}{2}\langle[\alpha,\alpha], \frac{\partial}{\partial\alpha}\rangle$, $\quad \Omega_p^{\beta} = 0$

$$\Omega^{\mathbb{A}\beta} = \langle \beta, F_-(\mathrm{ad}_{i\hbar\frac{\partial}{\partial\beta}})\mathbb{A}_p\rangle$$
$$\Omega^{\beta\mathbb{A}} = \langle \beta, F_+(\mathrm{ad}_{i\hbar\frac{\partial}{\partial\beta}})\mathbb{A}_p\rangle$$
$$\Omega^{\mathbb{B}\alpha} = \langle \mathbb{B}_p, F_+(\mathrm{ad}_{i\hbar\frac{\partial}{\partial\mathbb{B}_p}})\alpha\rangle$$
$$\Omega^{\alpha\mathbb{B}} = \text{-- -- --}\, F_- \,\text{-- -- --}$$

$F_+(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} + \ldots$

$F_-(x) = \frac{x}{1-e^{x}} = -1 + \frac{x}{2} - \frac{x^2}{12} + \ldots$

We have:
- $\Omega^2 = 0$
- $(\hbar^2\Delta_{res} + \Omega_\partial) Z = 0$ for any Σ with corners

- $(\mathcal{H}, \Omega)$ for a stratified circle disassembles into states on intervals and corners:

$\mathcal{H}_p^{\alpha} = \underset{\substack{\parallel \\ \Lambda \mathfrak{g}^*}}{\mathrm{Fun}(\mathfrak{g}[1])}$, $\Omega^{\alpha} = i\hbar\, d_{CE}$ — dg (com.) algebra, supercom. product

$\mathcal{H}_p^{\beta} = \underset{\substack{\parallel \\ S\mathfrak{g}}}{\mathrm{Fun}(\mathfrak{g}^*)}$, $*_\hbar$, $\Omega^{\beta} = 0$ — non-com. (dg) algebra

corner spaces of states — dg algebras.

BCH star-product $\quad e^{-\frac{i}{\hbar}\langle\beta,x\rangle} *_\hbar e^{-\frac{i}{\hbar}\langle\beta,y\rangle} = e^{-\frac{i}{\hbar}\langle\beta, BCH(x,y)\rangle}$

$\mathcal{H}^{\xi} \circlearrowright \mathcal{H}\left(\underset{\xi}{\overset{in}{\bullet}} \xrightarrow{\mathbb{P}} \underset{\xi'}{\overset{out}{\bullet}}\right) \circlearrowleft \mathcal{H}^{\xi'}$ — dg bi-module over $\mathcal{H}^{\xi}_{in}, \mathcal{H}^{\xi'}_{out}$

gluing of intervals $\longrightarrow$ tensor product of bimodules

$$\mathcal{H}\left(\underset{\xi_0}{\bullet}\overset{\mathbb{P}_1}{\text{———}}\underset{\xi_1}{\bullet}\overset{\mathbb{P}_2}{\text{———}}\underset{\xi_2}{\bullet}\right) = \mathcal{H}\left(\underset{\xi_0}{\bullet}\overset{\mathbb{P}_1}{\text{———}}\underset{\xi_1}{\bullet}\right) \underset{\mathcal{H}^{\xi_1}}{\otimes} \mathcal{H}\left(\underset{\xi_1}{\bullet}\overset{\mathbb{P}_2}{\text{———}}\underset{\xi_2}{\bullet}\right)$$